When Only the Market Can Vindicate You: Speculation and Inefficient Market Equilibria

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Abstract

A theory of bubbles and crashes in financial markets, as the market remains ignorant to new information and then suddenly adjusts to it. Those who possess the information have short-time evaluation constraints and will not trade if they do not expect that market prices will soon adjust to fundamentals. Their only chance is to submit the information to others via trading, which leads to a signaling game with multiple equilibria where the market is either informationally efficient or inefficient. In the presence of noise, there is also a coordination problem among informed speculators and changes in higher-order knowledge can lead to sudden equilibrium shifts.

Keywords: Speculation, informational efficiency of markets, multiple equilibria, market vindication, common knowledge, global games

JEL Classification: D82 (Asymmetric and Private Information), D84 (Expectations; Speculations), G14 (Information and Market Efficiency)

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> *Value is not only inherent in the stock; to do you any good, it has to be value that is appreciated by others. (Analysts at White, Weld walk around repeating "I have always preferred recognition to discovery" because that is an aphorism of one of the partners.)*
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> —Smith 1976, p. 212.

It is an unfortunate trait of financial markets that they can remain ignorant of important new information for a while and then suddenly and rapidly adjust to it. This can manifest itself in a sudden market crash after a bubble-like high, such as the Great Crash of 1929 or the bear market following the dotcom bubble.

This paper suggests a novel explanation for both phenomena – protracted ignorance and sudden adjustments – through a theoretical model. It takes as given that information arrival is gradual – new information is only known to a limited number of market participants. If these traders are subject to short-term evaluation constraints (as most investors are), they cannot be content to just buy and hold: If the market price does not adjust to their information (so that they can sell at a profit), it would be better for them to not trade on their information.

I propose that this can leave the market in an equilibrium where traders with a finite horizon do not trade on their information because they fear that the market will not adjust to it, and the market does not adjust to the information because no one is trading on it. When changes in the higher-order knowledge of traders occur that allow for coordination, a sudden shift to a more informationally efficient equilibrium is possible.

Since the analysis is qualitative, no inferences are drawn regarding quantitative phenomena such as volatility or liquidity. The contribution of this paper is in describing the mechanism in a theoretical model. Section 5 compares the model's setup and results to historical and recent episodes of speculative activity.

Consider a single asset that is traded in two periods, after which its fundamental value is realized. Some speculators know the fundamental value, but cannot hold the asset until the value is realized. There are long-term oriented, but ignorant investors in the market who would like to learn the fundamental value of the asset; but there is no viable direct channel through which the speculators could sell their information to others. Yet through the collective price impact of their trades in the first period, the speculators can transmit information about the fundamental value of the asset. They can, for example, buy if the value is high and then sell at a higher price to long-term investors in the second period, who afterwards profit from the high fundamental value. Then, in a way, the speculators have made use of the market to sell their information: Their profit is the difference between the initial market price and the price at which they can sell after trading, after other market participants have adjusted what they are willing to pay for the asset given that the speculators have bought.

The model contains two sources of random noise, which makes this information transmission harder. Firstly, there are noise traders who buy or sell randomly, making the overall order flow less controllable by speculators. Secondly, in period 1, speculators and noise traders trade not only with each other (as far as their trades offset each other), but also with some residual market consisting mostly of market-makers, who executes their remaining trades at some random market depth. A sufficient market depth, compared to the span of possible valuations, is needed so that speculators do not immediately drive the price to or beyond the fundamental value.

I show that there are two types of equilibria in the model, which stem from the signaling structure of information submission from speculators to the market. There is an *adjustment equilibrium* where speculators trade on their information if noise trading activity is low enough, and otherwise just follow the noise traders. Other market participants can then observe the movement in the market price, deduce some information about the value of the asset and adjust their beliefs accordingly. This change in beliefs leads to a further change in prices, after which speculators can eliminate their holdings at a profit. This adjustment equilibrium is (partially) informationally efficient in that the market price will often adjust to the speculators' information and only sometimes be driven by the noise.

In another, *non-adjustment equilibrium*, speculators never act on their information, and other market participants assume correctly that the market price is completely uninformative. This is similar to the "babbling equilibrium" of cheap-talk games, where the sender chooses a meaningless message and the receiver believes that the message is uninformative. This equilibrium is informationally inefficient and the market price is purely driven by noise.

In the adjustment equilibrium, the actions of the speculators are strategic complements, since every single speculator depends on other speculators making the same trade so that together they can move the market price. The situation is similar to a stag hunt, as speculators prefer to trade on their information as long as other speculators do the same, but otherwise would prefer to do the opposite. Because of this interdependence, it not only matters what each speculator knows, but also what he knows about the other speculators (because they make their choice based on their knowledge) and so on. In a final step, I assume that the strength of the noise trading in period 1 is not common knowledge among the speculators, so that it is not common knowledge whether there are enough speculators to push the market price in the right direction or not.[1] Then the speculators cannot coordinate on the adjustment equilibrium and the market price will not adjust to their information.

This paper takes up ideas and concepts from three different strands of literature: Short-term constraints and their influence in financial markets, circularity with self-fulfilling beliefs and equilibrium selection in multiple equilibria.

[1] "Common knowledge" is used in the sense of Aumann (1976), i.e. something is common knowledge if everybody knows that everybody knows and so on... that it is true.

Contrary to standard models of financial markets (e.g. Kyle, 1985, 1989; Glosten and Milgrom, 1985; Grossman and Stiglitz, 1980), informed traders have a short-term evaluation constraint in this model. Their choices are no longer strategic substitutes, but can be strategic complements (like the "positive spillovers" of Froot et al. (1992)). The assumption is not necessarily that there is only one generation of speculators which have information. Instead, every single speculator can also be thought of as an unfinished arbitrage chain of many speculators, similar to Dow and Gorton (1994).

Another feature is the dynamic circularity between prices and beliefs that even emerges in this simple two-period model: Trades move the market price, the price change is observed and leads others to update their beliefs; changes in beliefs lead to new trades which move prices. In this way, the paper contributes to the growing number of papers on circularity in financial markets, such as Hassan and Mertens (2011), Barlevy and Veronesi (2003) or Goldstein et al. (2013).

Together, complementarities and circularity lead to two different multiple-equilibrium structures. The overall model has a signaling structure, where speculators can submit their information through trading, but can just as well submit misinformation. Then there are two equilibria: Information is submitted and messages are believed, or no information is submitted and messages are ignored. Because there are many speculators acting at the same time, there are also multiple equilibria in the partial game in which only the speculators act, similar to other coordination games. Speculators can either all send the correct or the incorrect message.

The second multiple-equilibria structure is necessary to maintain the first one: The adjustment equilibrium in which speculators submit information is only possible if the speculators can coordinate among themselves to submit a message. This coordination, however, requires common knowledge among speculators about their strength relative to noise trading. Using the global games concept developed by Carlsson and van Damme (1993) similar to the application by Morris and Shin (1998), we can relax this assumption to see that the speculators cannot coordinate on sending any messages in the absence of common knowledge. It is thus speculators' higher-order knowledge that determines whether an equilibrium exists in which the market price adjusts to the speculators' information.

This paper has two novel contributions. Firstly, it shows that the signaling structure given by the short-term evaluation constraint leads to the existence of multiple market equilibria, and that the market can be stuck in an informationally inefficient equilibrium. Secondly, there is a coordination problem among those traders who bring new information into financial markets, and this can force the market to remain informationally inefficient. Methodically, I propose that the global games framework can be applied to a "partial game" played only by a subset of the players, which yields new insights on the role of coordination problems in the existence of equilibria.

1 The Model

1.1 General Structure

Consider the market for one asset. The asset has a present value v of either either v_H or v_L, where $v_H > v_L$ and both values are equally probable. There are two types of individual traders that are active in the market: *Speculators*, who have some information but a short horizon, and *noise traders*, who trade randomly. Additionally, there is a (residual) *market* which acts as a unified, utility-maximizing player but can be thought to consist of long-term investors, market makers and others.

The speculators receive information about the value v of the asset and can then act upon this information. To simplify matters, the speculators' information is perfect. But there are two sources of noise in the model: Firstly, there are noise traders that post buy or sell orders at the same time as the speculators. Secondly, after speculators and noise traders have submitted their orders, the market is cleared by uninformed and unsophisticated market makers according to a linear pricing function, where the exact market depth is unknown.

At the same time that the speculators receive information about v, they also become informed about the direction and size of overall order flow from noise traders. This seeks to capture the phenomenon that speculators not only have some information about the value of the asset, but that they can also observe the current market sentiment—and hence whether their private information is in line with this sentiment or not.

Trading occurs in two periods: In the first period, speculators and noise traders post buy or sell orders, which are executed with unknown market depth. In the second period, the speculators have to eliminate all their holdings and do this by trading with the market, which sets a price at which it is willing to buy or sell the asset. For this, the market tries to infer v from the first-period price p_1.

1.2 Assumptions in Detail

The Players There are infinitely many speculators and infinitely many noise traders. The speculators, who are perfectly informed about v, are ordered on the unit interval. They have a short horizon: They can buy or sell the asset in period 1, but need to liquidate their holdings (i.e. make the reverse trade) in period 2.[2]

The uninformed noise traders decide somehow (not necessarily on their own) whether to buy or sell one unit of the asset (because of liquidity needs, or irrational ideas about v).

[2] The assumption that there are infinitely many traders must not be taken literally - it is simply meant to represent the fact that traders do not take account of the price impact of their trades, or assume that they cannot influence the price. The model works just as well in a discrete setting with finite numbers of speculators and noise traders, see appendix E.

Denote by x_S the net order flow from the speculators, and by x_N the net order flow from noise traders in period 1. The order flow x_S from speculators is the result of their strategic trading decisions, while that x_N is simply the result of some random process. Specifically, we assume that x_N is distributed on some interval $[-n, n]$ where $n \in \mathbb{R}$ according to some distribution F that is symmetric around 0 (so that $F\left(x_N^{'}\right) = 1 - F\left(n - x_N^{'}\right)$).

We restrict our attention to cases where $n > 1$, i.e. there could potentially more trades from uninformed than from informed traders. Many noise trades will offset each other, so that the size of the net order flow from noise traders, $|x_N|$, is usually much smaller than n.

The speculators get a utility of $p_2 - p_1$ if they buy in period 1 and $p_1 - p_2$ if they sell in period 1, and 0 if they do nothing.

The Market Makers in Period 1 Let p_t be the market price of the asset in period t. At the beginning, the asset is trading at price p_0, which is the unconditional expected value of the asset:

$$p_0 = \frac{v_H + v_L}{2}. \tag{1}$$

This is equivalent to saying that the market in period 0 incorporates no private information about the value of the asset.

Let $x = x_S + x_N$ be the total order flow from speculators and noise traders in period 1. These orders will be cleared by some market makers according to the linear pricing function

$$p_1 = p_0 + \lambda x \tag{2}$$

where λ is an unknown reverse market depth parameter similar to Kyle's Lambda (Kyle, 1985). We assume that λ is uniformly distributed on the open interval $\left(0, \hat{\lambda}\right)$. To guarantee existence of well-behaved equilibria, we will impose a maximum condition on $\hat{\lambda}$ (i.e. a minimum condition on market depth) below.

The Market in Period 2 In period 2, the market is an intelligent player, who has to set a price p_2 at which it is willing to buy or sell any quantity of the asset. Since the market gets a utility of $-(v - p_2)^2$, it will always always maximize utility by setting $p_2 = \mathrm{E}[v \mid p_1]$. We can think of the market in period 2 as a large number of rational long-term investors, market makers and the like, who are in Bertrand-style competition and therefore make zero profit and are willing to buy or sell the asset for its expected value.

Restriction to Trade Size The speculators in period 1 can only buy or sell one unit of the asset each. The main intuition of this assumption is that the market is large compared to any single speculator. In the context of this model it is also a technically desirable assumption, since perfectly informed speculators with no trading or budget restrictions would otherwise have an incentive to trade arbitrarily large quantities and completely correct the price (as there is no fundamental risk for them). Just like in Glosten and Milgrom (1985), our focus is on the informational content of trades, not on their size.

All traders (speculators, noise traders and the market) are free of inventory considerations. Speculators can either buy one unit in period 1 and then sell it in period 2, or sell in period 1 and buy back in period 2, or they can abstain from trading at all. Selling and later buying back can also be thought of as a short sale (which has an inherent short horizon, even if we were to assume that speculators were not short-term interested). The market in period 2 is willing to trade any number of units at a fixed price.

Perfectly informed Speculators The speculators in this model are "market insiders" who are perfectly informed not only about the true value of the asset, but also about how the noise traders are (overall) trading. We could imagine that they can compare market sentiment and perhaps even some market movements (although this model is static) with the real value v, which they know.[3] Formally, the speculators learn v and x_N at the beginning of period 1.

The speculators do not know inverse market depth λ, the other source of noise in the model. The noise in λ mostly serves to reduce the informativeness of p_1 such that p_1 doesn't fully reveal who has been trading in which direction (and thus reveal v). The fact that speculators don't know λ also precludes the existence of equilibria in which the speculators submit information by precisely encoding it into the price.

The market only knows the probability distributions of v and x_N, and observes p_1 at the beginning of period 2 before deciding which price to offer.

Timing of the Model The explicit timing of the model is shown in figure 1.

While the market behaves rationally in using all information that is contained in p_1, it is conceivable that it could also condition on order flow in period 2 when speculators liquidate their holdings. In particular, it could act similar to the market makers of Glosten and Milgrom (1985) and adjust p_2 conditional on whether it receives buy or sell orders. But the assumption that all speculators liquidate their holdings at in period 2 is merely a simplification. In reality, many or even most traders are short-term oriented not because they have to liquidate their holdings every few days or weeks, but because their holdings get evaluated, by themselves or their superiors, *at market prices* in short time intervals. For their motivation and strategic choice, this is equivalent to a world in which they had to completely sell off and rebuild their portfolio frequently—but it does not per se allow the investors in our model to deduce any

[3]Cf. the discussion on insider trading by Leland (1992), who works with a similar assumption, and the "private learning channel" that speculators have in Cespa and Vives (2012).

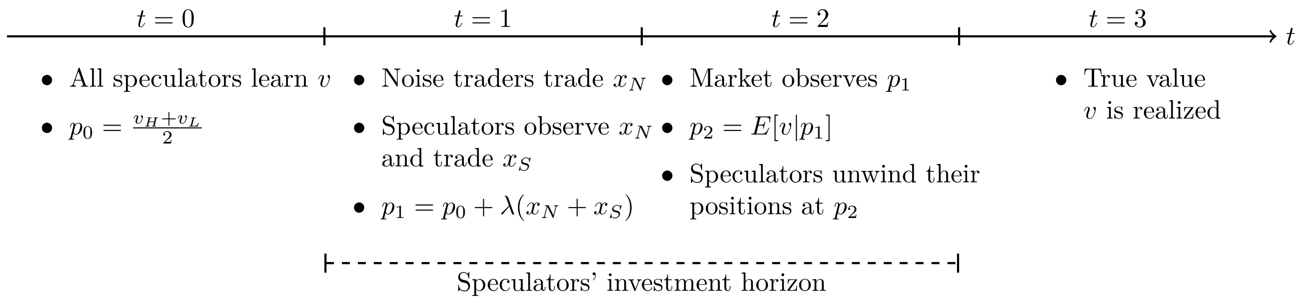

Figure 1: The timing of the model. The dashed line shows the speculators' investment horizon, which does not stretch to the realization of fundamental value in period 3 as speculators need to unwind their position in period 2.

information about the order flow from the orders they face in period 2.

This model has two types of equilibria, which are explored in each of the following two sections.

2 The Adjustment Equilibrium

In this equilibrium, the market in period 2 assumes that p_1 is informative about v. In particular, it assumes that if $p_1 > p_0$, it is more likely that $v = v_H$ and vice versa. p_2 is set accordingly. If $|x_N| < 1$, the speculators can therefore influence p_2 by their trading, and they buy if v_H and sell otherwise. If $|x_N| \geq 1$, however, whether p_1 is above or below p_0 is determined by the direction of the noise trading, and speculators cannot influence p_1 sufficiently. They therefore just trade in the same direction as the noise traders.

The market adjusts its expectation of v according to the function $p_2(p_1)$, which takes the behavior of the speculators and the distribution of x_N into account. If $p_1 > p_0$, for example, they know that overall order flow in the first period was positive, and that therefore either $|x_N| < 1$ and $v = v_H$, or that $|x_N| \geq 1$ and the speculators just followed the herd. The existence of the equilibrium is assured by a maximum condition on inverse market depth, which guarantees that it will always be optimal for the speculators to follow their equilibrium strategy.

Proposition 1. *(Adjustment equilibrium) It is an equilibrium if every speculator follows the strategy "If $x_N \leq -1$, sell and if $x_N \geq 1$ buy. If $|x_N| < 1$, buy if $v = v_H$ and sell if $v = v_L$." and the market sets*

$$p_2 = p_2^H(p_1) = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L \quad \text{if} \quad p_1 > p_0 \tag{3}$$

$$p_2 = p_2^L(p_1) = (1 - \pi(p_1))\, v_H + \pi(p_1)v_L \quad \text{if} \quad p_1 < p_0 \tag{4}$$

$$p_2 = p_0 \quad \text{if} \quad p_1 = p_0, \tag{5}$$

where

$$\pi(p_1) = \begin{cases} \frac{1-F\left(\frac{p_1-p_0}{\hat{\lambda}}-1\right)}{2-F\left(\frac{p_1-p_0}{\hat{\lambda}}-1\right)-F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}}-1,1\right\}\right)} & \text{if } p_1 > p_0 \\ \\ \frac{1-F\left(\frac{p_1-p_0}{\hat{\lambda}}+1\right)}{2-F\left(\frac{p_1-p_0}{\hat{\lambda}}+1\right)-F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}}+1,1\right\}\right)} & \text{if } p_1 < p_0 \end{cases}$$

is the market's belief that $v = v_H$ if $p_1 > p_0$ or that $v = v_L$ if $p_1 < p_0$, respectively, if and only if

$$\hat{\lambda} \leq E\left[\frac{F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}}-1,\,1\right\}\right)-F\left(\frac{p_1-p_0}{\hat{\lambda}}-1\right)}{4-2F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}}-1,\,1\right\}\right)-2F\left(\frac{p_1-p_0}{\hat{\lambda}}-1\right)}\middle| x_N = n\right]\frac{(v_H-v_L)}{n+1}. \tag{6}$$

Proof in Appendix A.

The intuition of the proof is the following: If speculators follow their equilibrium strategies, p_1 will contain some information about v. The function $\pi(p_1)$, which takes account for the distributions of x_N and λ, gives the probability (and hence the equilibrium belief of the market) that $v = v_H$ for every $p_1 > p_0$, and the probability that $v = v_L$ for every $p_1 < p_0$. Then the equations (3) to (5) give the expected value of v for different p_1. Since all possible prices occur in equilibrium, we do not need to consider out-of-equilibrium beliefs.

The speculators, on the other hand, will make an expected profit by following their equilibrium strategies, since the price movement in period 1 is always small enough (if market depth is sufficient, which is where the maximum condition on $\hat{\lambda}$ comes from) so that p_1 is lower than $p_2^H(p_1)$ and higher than $p_2^L(p_1)$. Because a single speculator has only limited influence on p_1, no single speculator has an incentive to deviate. If a speculator would deviate from his equilibrium strategy, he would make a loss equal to the profit of his equilibrium strategy.

(6) may not immediately look like a maximum on $\hat{\lambda}$, since $\hat{\lambda}$ actually appears on both sides of the inequality. But since $\frac{p_1-p_0}{\hat{\lambda}} - 1 = \frac{\lambda}{\hat{\lambda}}(x_S + x_N) - 1$, and λ is distributed uniformly on $(0, \hat{\lambda})$, the distribution of the term $\frac{p_1-p_0}{\hat{\lambda}} - 1$ is actually independent of the size of $\hat{\lambda}$. Therefore, the right-hand side of (6) is constant in $\hat{\lambda}$.

The maximum condition on $\hat{\lambda}$ guarantees that for any x_N, all speculators have an incentive to trade. It has the form $\hat{\lambda} \leq \phi\frac{v_H-v_L}{n+1}$. We can understand this in the following way: $\frac{v_H-v_L}{n+1}$ means that, even if all orders should have the same direction, p_1 will in expectation not leave the interval (v_L, v_H). This is then multiplied by a correction factor ϕ to take account of the precise shape of F, according to which the market updates its conditional price $E[v|p_1]$. This correction factor ϕ is in $(0, 1)$: It is always larger than 0, since for $x_N = n$ there exists some small λ so that for $p_1 = p_0 + \lambda(x_N + 1)$ it is $F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) > F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)$. And it is always smaller than 1, since for $x_N = n > 1$ there is no λ such that $F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) \geq 1$.

Figure 2 shows an exemplary price path in the adjustment equilibrium, where noise is small.

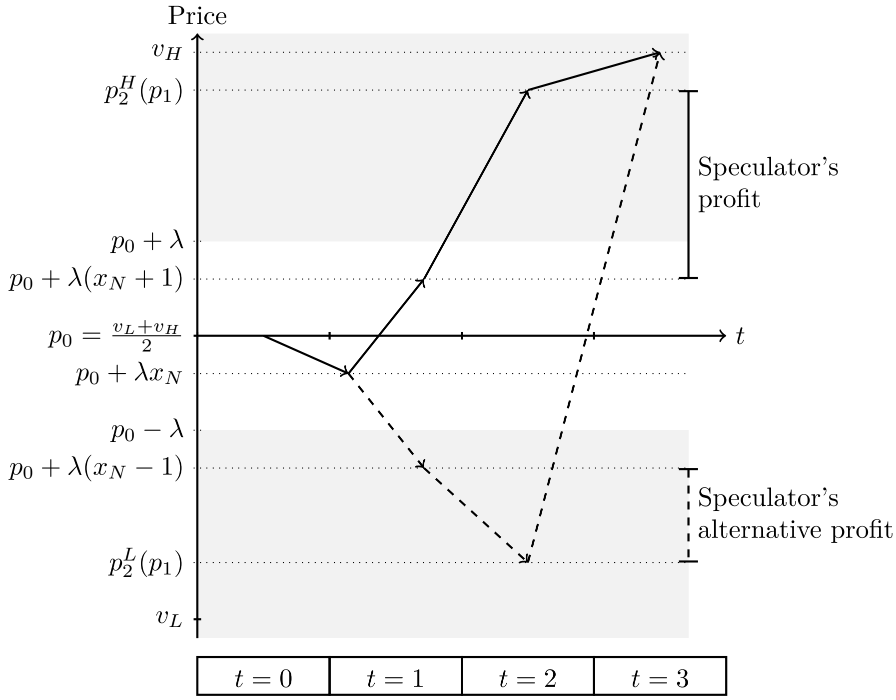

Figure 2: The equilibrium price path in the adjustment equilibrium for a given set of parameters, where $v = v_H$ (asset value high) and $0 > x_N > -1$ (noise traders sell the asset). Noise trading is small, i.e. noise traders do not push the price outside the white area in the center. All speculators buy the asset, thus pushing the price to $p_1 = p_0 + \lambda(x_N + 1)$. The market observes $p_1 > 0$ and sets $p_2 = p_2^H(p_1)$, so that speculators make a profit. In period 3, the true value v_H is realized. Note that at $t = 1$, the speculators could coordinate on selling the asset instead, driving the price to $p_0 + \lambda(x_N - 1)$ and also making a profit. This would constitute an equilibrium of the speculators' coordination game, but could not be part of a subgame-perfect equilibrium, however, since the market would be better off by believing that $E[v|p_1 < p_0] > p_0$ instead.

The speculators then face a coordination game: They can either all buy or all sell, which will place p_1 either above or below p_0. In both cases they make a profit, and both cases constitute an equilibrium of their coordination game. In the subgame-perfect equilibrium, however, the market must optimally extract information from p_1, which is only the case if speculators trade towards the fundamental value v.

If $|x_N| \geq 1$ the speculators cannot influence whether p_2 will be above or below p_1, since they cannot neutralize the noise trading and there is no coordination game among them. It is dominant for them to follow the herd—regardless of whether it is right or wrong. If the noise traders are wrong, that means that the speculators will drive the market price further away from its correct value even though they know better, and even though the investors would gladly enrich them by buying the asset at a more correct price. Figure 3 shows such a price path. Since the noise trading is now large, the price gets pushed too far away from p_0 (into the grey area), so that the speculators are not able to move it above p_0 again. Once noise trading has

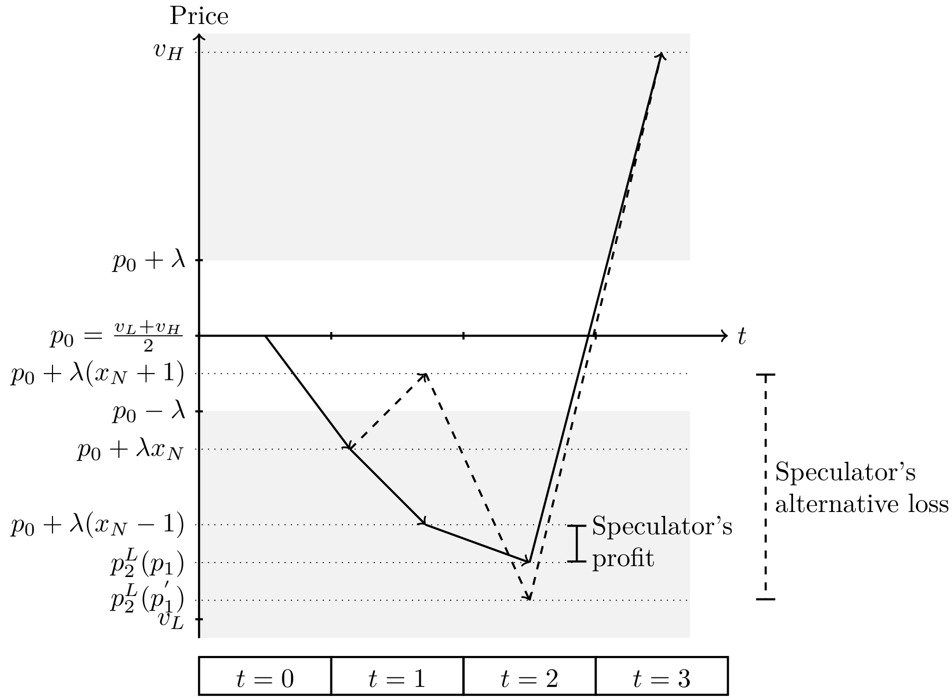

Figure 3: Another equilibrium price path in the adjustment equilibrium. Now $x_N < -1$, i.e. noise traders push the price into an area (given by the gray shade) where it is the dominant action for speculators to sell. Unlike in the previous figure, the coordination game between speculators does not have multiple equilibria and thus no information submission is possible.

pushed the price into the "grey area" of the graphs, p_1 and p_2 will not return to the informative "white area" and are therefore uninformative.

De Long et al. (1990) describe a similar effect when they write about "noise trader risk": In their model, rational and informed arbitrageurs in an overlapping-generations model could correct mispricings that arise through noise trading. But since arbitrageurs are short-lived and the market could get even more irrational (noise trade is randomly distributed), they refrain from fully correcting the mispricings. In my model, the direction of the noise order flow (and therefore also the direction of the mispricing in the next period) is known to the speculators, and they can therefore choose to trade against their information and therefore avoid the noise trader risk. They drive prices further away from fundamentals while doing so, as in models of speculative bubbles such as Abreu and Brunnermeier (2003).

3 The Non-Adjustment Equilibrium

Besides the partially efficient equilibrium, there is also an informationally inefficient equilibrium.

Proposition 2. *(Non-adjustment equilibrium) It is an equilibrium if speculators with probability* $\min\{|x_n|, 1\}$ *either buy if* $x_N < 0$ *or sell if* $x_n \geq 0$*, and the market believes that* p_1 *is completely*

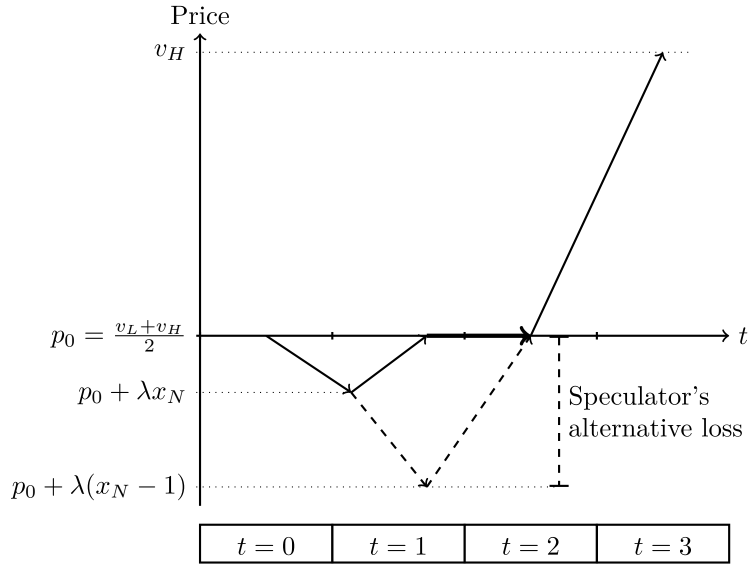

Figure 4: The equilibrium price path in the non-adjustment equilibrium for a given set of parameters. Since the market assumes p_1 to be uninformative, $p_2 = p_0$ and the speculators' optimal actions are independent of v: They simply act as trading partners for the noise traders. p_1 thus becomes uninformative about v. In the case shown here, $p_1 = p_0$ and speculators make no profit; if $|x_N| > 1$ speculators' profit is given by $|p_1 - p_0|$.

uninformative and therefore sets $p_2 = p_0$.

If the market believes p_1 to be uninformative, the speculators already know that $p_2 = p_0$ and the only gain they can make is by providing liquidity to noise traders. Since this means they do not act on their information about v, the market is correct to believe that p_1 is uninformative.

Proof. Assume that the market follows this strategy, so that $p_2 = p_0$. It is then profitable for any speculator to buy at $p_1 < p_0$ or sell at $p_1 > p_0$. Speculators therefore trade against the noise traders until either all of them have posted an order or $x = 0$ and $p_1 = p_0$. No speculator has any incentive to deviate: Those who post orders either make a positive profit (if $|x_N| > 1$) or no profit (otherwise), and those who do not post orders (since other informed speculators have already driven the price back to p_0) would lose money by trading (since they would move the price above p_0 if they bought or below p_0 if they sold).

Now assume that the speculators follow this strategy. Then p_1 contains absolutely no information about v, since the speculators only either do nothing or counteract the noise traders (whose actions are independent of v), and none of their behavior is conditional on v. The market can therefore only follow its prior and set $p_2 = p_0$. □

Figure 4 shows the price path in the non-adjustment equilibrium. Note that there is no similar coordination problem among speculators as there is in the adjustment equilibrium.

In this equilibrium, total order flow x will be between $1-n$ and $n-1$, meaning that $p_1 \in \left[p_0 + \hat{\lambda}(1-n),\, p_0 + \hat{\lambda}(n-1)\right]$. Out-of-equilibrium beliefs—what the market thinks if p_1 should lie outside that interval—should therefore be examined. But it is clearly not optimal for the market to assume that prices outside this interval are informative. If it did, and accordingly set some $p_2 > p_0 + \hat{\lambda}(n-1)$ after observing $p_1 > p_0 + \hat{\lambda}(n-1)$, the speculators would have an incentive to try to push p_1 above $p_0 + \hat{\lambda}(n-1)$ regardless of whether $v = v_H$ or $v = v_L$, so that p_1 would not be any more informative than it was before.[4]

No player has an incentive to deviate from their equilibrium strategies: The market would not benefit from assuming that prices contain information, and the speculators cannot gain from unilaterally submitting information (and thereby driving p_1 away from p_0). In this interplay of "not talking" and "not listening", the equilibrium is similar to the "babbling equilibrium" of cheap-talk games (Farrell and Rabin, 1996). There, the sender randomizes between messages such that her message has no correlation to her private information, and the receiver ignores any messages by the sender. This constitutes an equilibrium, albeit (when it comes to everyday communication) perhaps not a plausible one.

Here, speculators act as senders, and they can only transmit information about the value of the asset through the noisy market price. If the market, as receiver, picks up the signal, it will offer a price which makes the whole endeavor profitable for the speculators, and they play the efficient equilibrium. The randomization device used in the uninformative equilibrium is the behavior of the noise traders, which has no correlation to the private information of the speculators about the asset value.

We can imagine that a market gets stuck in the non-adjustment equilibrium, i.e. that it is played many consecutive times in a dynamic game, with informed speculators and serially fluctuating noise. The price of the asset, set by the uninformed market and pushed upwards or downwards by noise, would remain stationary; speculators would make money from providing liquidity to noise traders but not making use of their knowledge about v. This idea will be taken up again in the discussion below.

The models of Froot et al. (1992), who describe "positive information spillovers", and Hellwig and Veldkamp (2009) who describe strategic complementarities in markets, have similar aspects of "it is only worth knowing what others know" and "it is only worth trading on information that others also trade on". Here, however, the effects arise from a different setting that has neither the randomized timing of Froot et al. nor the beauty-contest structure of Hellwig and Veldkamp. In particular, there is no information choice here, but the multiple equilibria arise from a given information structure through the trading mechanism.

[4] If, on the other hand, they were to set p_2 with $p_0 < p_2 < p_0 + \hat{\lambda}(n-1)$ after observing $p_1 > p_0 + \hat{\lambda}(n-1)$, the speculators would have no incentive to drive prices out of equilibrium range at all, even if they could submit information in this way.

4 The Coordination Problem of the Speculators

4.1 The Speculators' Problem as a Reduced-Form Game

Now consider the problem of the speculators in the adjustment equilibrium. Clearly, the speculators need to coordinate: If $v = v_H$ and $x_N < 0$, enough speculators need to buy the asset to push p_1 on the right side of p_0 so that the market can draw correct conclusions. If we take the strategy of the market as given and assume that the market behaves as in the adjustment equilibrium, we can analyze the reduced-form game that the speculators play among themselves in period 1. This game is not supermodular, but the coordination aspect still leaves several equilibria. It is an equilibrium of the partial game if all speculators play their equilibrium action – but also, if $|x_N| < 1$, if they all do the opposite and work together to "trick" the market.

A more detailed analysis of this reduced-form game can be found in appendix C. Here it suffices to know that (a) the speculators have a coordination problem, (b) their optimal action is dependent on the actions of other speculators and (c) in the partial game played by the speculators if the market believes that p_1 is informative, there are two pure-strategy equilibria, one of which corresponds to the equilibrium action of the adjustment equilibrium and the other "tricks" the market into a wrong belief (and could therefore not be part of a perfect equilibrium of the overall game). It is this multiplicity that allows for information submission.

4.2 The Adjustment Equilibrium without Common Knowledge

The multiplicity of equilibria in the coordination game among speculators exists if all parameters that are known to any speculator are also common knowledge among the speculators. Is is this common knowledge that allows them to coordinate on trading in the direction of fundamental value, which makes the adjustment equilibrium possible.

In reality, however, both v and x_N are likely to not be common knowledge. They are information that the speculators gain through private learning channels, by their own research or observation or through private communication. In the absence of such common knowledge, coordination gets harder, as speculators start worrying about what other speculators believe, what they believe about others' beliefs, and so on.In this model, the main ingredient of the speculators' coordination game is the knowledge of x_N, i.e. the speculators' knowledge on whether they actually can influence the future price to a sufficient degree.[5] If we relax this common knowledge assumption, the model will mirror the realistic effect that speculators consider the beliefs of others before making their trading decision—an effect which is absent from the complete-information model.

[5] Note that the mass of speculators is normalized to 1, and that x_N therefore simply is the relative strength of noise trading.

Assume that every speculator now observes some ω, which is independently drawn from a uniform distribution on $[x_N - \varepsilon, x_N + \varepsilon]$, i.e. an interval of 2ε around the true x_N, with $\varepsilon > 0$ but small. Every single speculator will then know after observing ω that $x_N \in [\omega - \varepsilon, \omega + \varepsilon]$. But about the beliefs of another speculator he will only know that the other speculator believes that $x_N \in [\omega - 2\varepsilon, \omega + 2\varepsilon]$, and so on. Then, even if the observation of every single speculator is extremely precise, it is only common knowledge that $x_N \in [-n, n]$—which was already known before (since the overall game structure is still common knowledge). Now the two equilibria of the speculators' coordination game no longer exist.

Proposition 3. *There is a unique equilibrium where all speculators buy if they observe* $\omega \geq 0$ *and sell otherwise. Proof in appendix A.*

The proof is similar to the proof of theorems 1 and 2 in Morris and Shin (1998), who build on the seminal framework by Carlsson and van Damme (1993).

Intuitively, if the market assumes that p_1 is informative and sets p_2 accordingly, but x_N is not common knowledge among the speculators, every single speculator has to reason along the following lines:[6]

> I know that x_N is within a small interval around my observation ω. If ω is in $(-1, 1)$, it is my best guess that all speculators together could overcome the noise so that p_1 correctly reflects our private information. I also know that the other speculators have a very precise idea about x_N—but my knowledge about their knowledge is a little less precise than my own knowledge about x_N. If I consider my knowledge about their knowledge about my knowledge, it gets even less precise.
>
> In particular, if ω is very close to 1, I think it is very likely that many other speculators have received a signal above 1 and will therefore play what they believe is the dominant strategy of buying. So I should also buy if I observe ω very close to but below 1.
>
> Now, the others will reason the same, so that if I observe ω somewhat less close to 1, I know that many others will observe a ω closer to 1, and buy for the reason outlined above. Such infection carries on, and vice versa from ω close to -1. So I will choose to sell if $\omega < 0$ and buy if $\omega \geq 0$, and disregard my private information about v.

Figure 5 depicts the intuition of the infection argument in a graph similar to the ones above.

If the trading of all speculators is only dependent on ω and independent of v, p_1 will actually be completely uninformative about v. Recall that we started out with the assumption that the

[6] Note that this is using a rationalizability approach, making use of the infection argument of Carlsson and van Damme (1993) or Rubinstein (1989)—not exactly following the reasoning of the proof. Since there is a unique rationalizable outcome, it is also the unique Nash equilibrium.

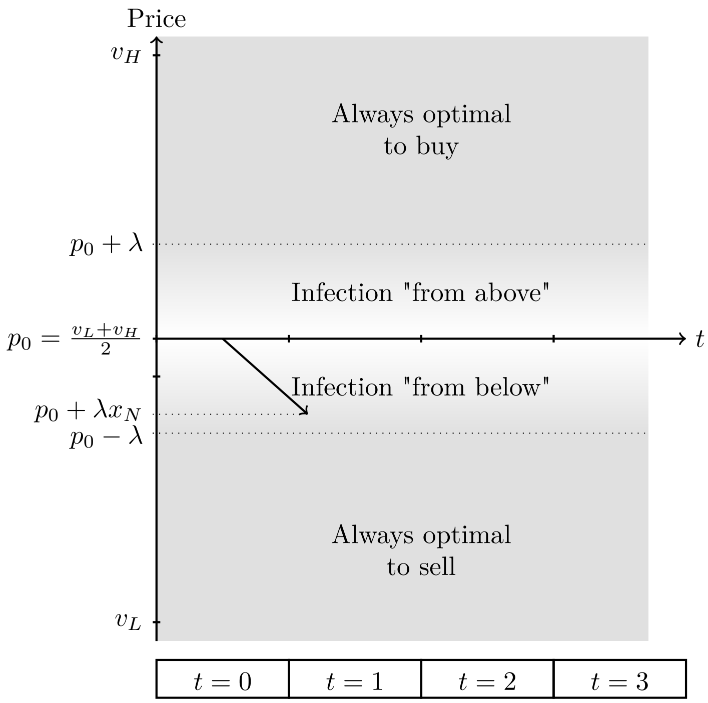

Figure 5: Infection in the absence of common knowledge. Even though $x_N > -1$, the speculators' coordination game no longer has multiple equilibria. A speculator observing $x_N > -1$ but close to -1 is worried that others might believe that $x_N < -1$, or that others might believe that others believe this, etc. This infection carries on, such that selling becomes optimal for all $x_N < 0$ and buying for all $x_N > 0$.

market believes that p_1 is informative—now we have seen that if the market reacts to p_1 as if it contained information, the speculators will behave in a way so that p_1 is actually uninformative. Hence there is no adjustment equilibrium if x_N is not common knowledge among the speculators.

4.3 The Non-Adjustment Equilibrium also exists without Common Knowledge

Now consider the non-adjustment equilibrium in which speculators trade against the noise and the market sets $p_2 = p_0$. The strategies of the speculators are pure strategical substitutes—there is no complementarity, as there is in the efficient equilibrium.

If each speculator can only observe his signal ω, it is still optimal to buy if $\omega \leq -1$, because in expectation $p_1 < p_0$ regardless of the behavior of other speculators. Now consider the case where $\omega \in (-1, 0)$. If all other speculators buy with probability $-\omega$ upon observing $\omega \in (-1, 0)$, they will on average buy with probability $-x_N$, which means that p_1 will be 0 in expected terms. Every single speculator is then indifferent between buying or selling or doing nothing. Therefore, it is an equilibrium if all speculators buy for $\omega \leq -1$, buy with probability $-\omega$ for $\omega \in (-1, 0)$, sell with probability ω if $\omega \in (0, 1)$ and always sell if $\omega \geq 1$. The non-adjustment equilibrium remains completely undisturbed if x_N is no longer common knowledge.

5 The Empirical Relevance of the Model

5.1 The Assumptions of the Model

A central assumption of this model is the information arrival, where new information is only learned by short-term speculators. That does not mean that *all* information arrival at financial markets works in that way – just that the model is concerned with the working of markets where this is the case. In general, however, it does not seem a wholly unreasonable assumption that speculators could be better informed than some long-term investors. Just consider that most professional money managers would count as "speculators" in the context of this model if we consider sufficiently long time periods—a few weeks, say, or a quarter. Few of them are allowed and capable of raking up massive losses over such a time frame even if they claim to have superior knowledge that will in the end be vindicated.

Empirical evidence suggests likewise that a large proportion of stock positions are opened for a very limited amount of time, with the expectation of making a profit in less time that it takes to see two quarterly earnings reports. The average holding period of stocks in the United States is three to four months—not even enough to receive a full dividend payment, let alone profit from long-term business or macroeconomic developments.[7] And even where assets are not bought and sold within days or mere seconds, those who decide about trading them have their performance evaluated at market prices at very short intervals. If a trader buys an asset at time t for the price p_t, it does not matter to him whether he sells the asset at $t+1$ and it contributes $p_{t+1} - p_t$ to his cash holdings, or whether he still holds it at $t+1$ and it contributes $p_{t+1} - p_t$ to the overall appreciation of his holdings since t.

Another crucial assumption is the presence of considerable noise – both in the form of random noise traders, but also in the form of random market depth. Some authors (e.g. Dow and Gorton, 1994, p. 825) argue that the presence of noise traders has to be explained. But the absence of noise traders would mean that all traders, at all times, act rationally to maximize their expected utility. There are two main types of traders for whom that does not apply. Firstly, substantial research on behavioral finance has shown that traders, institutional or private, fall prey to a large number of irrational biases. Secondly, even a rational trader might find it optimal to sell an asset whose price he expects to rise for liquidity reasons—for example when he needs to access his savings to retire or pay unforeseen expenses In any case, the assumptions about noise are supposed to illustrate the problems that speculators have when trying to transmit information through the market. As such, they also serve the purely technical purpose of obscuring the precise actions of any small group of speculator—an obscuring that undoubtedly also happens in real markets.

Finally, the signaling structure that is at the core of the existence of different market equilibria relies on the assumption that other market participants watch market prices to gain fundamental information. We have seen in the model that it is also rational for them to do so, as prices can indeed be very informative.[8] Watching for signals of informed trading by "small" market participants can be a profitable strategy, especially since large investors are probably not in Bertrand competition as the technical assumption in this model suggests.

Consider also the findings by Ljungqvist and Qian (2014), who analyze the similar problem of "shallow-pocketed arbitrageurs" who need to transmit their information to long-term investors so that prices adjust. While the mechanism they consider is different from the one modelled here (and perhaps easier to verify in practice), the underlying problem is similar.

[7]The "World Bank Financial Development Indicators" show stock market turnover ratios, which is the inverse of average holding period. In the United States in 2008, for example, trade volume was 4.35 times as high as total market capitalization.

[8]As a matter of fact, Chen et al. (2007) can show that managers even watch stock prices to gain information about the prospects of *their own* company.

5.2 Multiple Equilibria and Equilibrium Switches

As an example of a huge apparent mispricing in history, consider internet stocks in the late 1990s and early 2000s.[9] Many sophisticated fund managers, including Stanley Druckenmiller of the multi-billion dollar Quantum Fund, knew that internet stocks were overvalued. But they could not coordinate on selling off, and therefore it would have been foolish for any of them to start getting out of the market while prices were still rising.[10] In the context of the model, we could think of internet stocks being either worth v_L ("most of these companies will never make a profit") or v_H ("they will change the economy forever"). Most market participants don't know which is the case, but because v_H is so extremely large their unconditional prior $\frac{v_L+v_H}{2}$ is also large. Some sophisticated speculators know that $v = v_L$, but the rest of the market is not listening, and they cannot coordinate on selling off at the same time. Therefore, selling (or short selling) against positive noise order flow would lead to expected losses in the short-run, and the informed speculators refrain from it and concentrate on providing liquidity, and participating in the bubble, instead. Thus the market ends up playing the non-adjustment equilibrium.

At other times, speculators trade without considering the short-term evaluation constraint, and later pay dearly for it. The hedge fund Long-Term Capital Management (LTCM) began in 1997 to bet against the spread between Royal Dutch and Shell stocks. The stocks were trading at different exchanges, but prices should have been at a fixed proportion, because cash flows were paid in a fixed proportion. Instead, there was an 8% spread against which LTCM bet $2.3 billion. It was not controversial at the time that the spread should probably narrow at some point, so LTCM expected that others would follow in buying against the spread. These others could have been doing that either because they learned from observing LTCM's actions (as the market in the model) or because they expected others to learn (like the speculators of the model). But apparently, the unfoundedness of the spread was not common knowledge, and no others stepped forward—a fact that greatly surprised and "mystified" the managers at LTCM.[11]

If there are situations where the market "plays" the non-adjustment equilibrium, we should be observing sudden equilibrium shifts when changes in higher-order knowledge occur—when common knowledge is generated where before there was only finite-order-knowledge.

Are there large price movements that were not directly the consequence of the arrival of new fundamental information? In fact, Cutler et al. (1989) show that none of the 50 largest price movements of the S&P between 1945 and 1989 followed any obvious piece of news.

For an example outside the scope of that study, consider the timing of events surrounding

[9] For example the discussion in Abreu and Brunnermeier (2003, p. 175), where the authors argue that the bubble was a matter of timing—which it surely also was.

[10] Indeed, Abreu and Brunnermeier discuss the cases of two managers who had to resign precisely because they left the technology market when it took off.

[11] Cf. Lowenstein (2000, p. 148). The spread would eventually widen to 22%, contributing to LTCM's collapse.

"The Great Crash" of 1929. Financiers and traders had gradually come to realize that the exceptionally high share prices in the fall of 1929 were not sustainable.[12] But (short-) selling did not seem attractive, since prices kept rising. And the informed speculators—those who believed prices were too high—lacked common knowledge of the fundamental weakness of shares. Every single trader may have thought about selling, but feared that no other sophisticated speculators would do the same, and that unsophisticated investors would fail to read the information out of market prices and act upon it.

But on October 24 ("black thursday"), prices fell suddenly and violently by nearly 13%. They swiftly recovered (the closing was only 2.1% down that day), but the event had generated common knowledge about the weakness of market prices. In the following days, despite no substantial economic news (cf. Shiller, 2000, p. 94), informed market participants could now coordinate on selling, and the Dow fell over 23% in two days.

In the more recent Irish crisis, it had gradually become clear throughout 2008 that banks, entangled in the housing bubble, were not healthy. In September 2008, a wide-ranging guarantee by the Irish government had seemingly stabilized the banks' share prices. But when the head of the Financial Regulator appeared on the popular news show "Prime Time" on October 2, he claimed that bad lending was not responsible for the crisis, and that Irish banks were not in much trouble at all.[13]

"Prime Time" is popular enough in Ireland to serve as some kind of common-knowledge-generating device, especially among the group of people likely to be interested in such topics. So through this airing, the commonly known weakness of the banks as well as the inappropriate response by the regulator was made common knowledge, leading to a sudden equilibrium switch. In the following three days, share prices of Irish banks fell rapidly: Bank of Ireland lost almost 30%, Irish Anglo Bank about 50%, and Allied Irish Banks more than 25% of their value.

Traders often learn new information from conversations, internal reports, or less well-known media outlets. This gives them knowledge, perhaps even some finite higher-order knowledge about what others know. But only if something is reported in a common-knowledge source does it suddenly become common knowledge, by an infinite stacking of higher-order knowledge. Such common knowledge is an indispensable prerequisite for the existence of the informative equilibrium.

[12] Cf. Galbraith (1954), ch. 2 for the uneasiness in regulatory circles and attempts to deflate the bubble, and pp. 72–74 for prescient warnings by well-known bankers, financial services and the *New York Times*.

[13] See Lewis (2012, pp. 97–98) for an entertaining account of the episode.

6 Conclusion

Perhaps the main reason for the triumph of capitalism is that no other mechanism can transmit information about scarcity, efficiency and ability as realiable, fast and cheap as the price mechanism (cf. Hayek, 1945). We live in a system of financial capitalism because financial markets are the ultimate way of transmitting information: Financial assets are standardized and fungible, all information other than prices is stripped away, information flow is immediate and transaction costs minimal. But it is of critical importance to understand how well financial markets perform at aggregating information, and under what conditions they might not do well.

The role of coordination motives, and hence higher-order beliefs, has been found to be substantial in markets with a feedback loop between fundamentals and prices, such as foreign exchange Obstfeld (1996); Morris and Shin (1998) and bank runs Diamond and Dybvig (1983); Rochet and Vives (2004). The model in this paper suggests that higher-order beliefs of financial market participants and common knowledge generation, for example through mass media, can play a large role in all financial markets and can determine whether a market is informationally efficient or not.

Appendix

A Proofs

Proof of Proposition 1:

Part 1: The market has no incentive to deviate (and $\pi(p_1)$ is the correct belief).

Assume that the speculators follow their equilibrium strategies and consider the case where $p_1 > p_0$. The market can then, from observing p_1, draw conclusions about v. Let $\pi(p_1)$ be the conditional probability that $v = v_H$ after observing a certain p_1, $\Pr(v_H|p_1)$.

It is

$$\begin{aligned}\pi(p_1) &= \Pr(v_H|p_1) = \frac{\Pr(p_1 \cap v_H)}{\Pr(p_1)} \\ &= \frac{\Pr(p_1 \cap v_H \cap |x_N| < m) + \Pr(p_1 \cap v_H \cap x_N \geq m)}{\Pr(p_1 \cap |x_N| < m) + \Pr(p_1 \cap x_N \geq m)} \\ &= \frac{\int_{-1}^{n} g\left(\frac{p_1 - p_0}{x_N + 1}\right) dF(x_N)}{\int_{-1}^{n} 2^{\mathbf{1}_{x_N > 1}} g\left(\frac{p_1 - p_0}{x_N + 1}\right) dF(x_N)}\end{aligned}$$

where g is the density of λ. Since $g\left(\frac{p_1 - p_0}{x_N + 1}\right) = \frac{1}{\hat{\lambda}}$ if if $0 < \frac{p_1 - p_0}{x_N + 1} < \hat{\lambda}$ and 0 otherwise, we can rewrite this as

$$\begin{aligned}\pi(p_1) &= \frac{\int_{\frac{p_1 - p_0}{\hat{\lambda}} - 1}^{n} dF(x_N)}{\int_{\frac{p_1 - p_0}{\hat{\lambda}} - 1}^{n} 2^{\mathbf{1}_{x_N > 1}} dF(x_N)} \\ &= \frac{1 - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - 1\right)}{2 - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - 1\right) - F\left(\max\left\{\frac{p_1 - p_0}{\hat{\lambda}} - 1, 1\right\}\right)}\end{aligned}$$

$\Pr(v_L|p_1)$ is the complementary probability $1 - \pi(p_1)$, so that the expected value of v given p_1 is $E[v|p_1] = \pi(p_1)v_H + (1 - \pi(p_1))v_L$. A similar argument applies to the case where $p_1 < p_0$. If $p_1 = p_0$, the price contains no information and p_2 should be set equal to the prior.

p_1 is between $p_0 - \hat{\lambda}(n + 1)$ and $p_0 + \hat{\lambda}(n + 1)$. For $x_N \in \{-n, n\}$, all possible p_1 occur with positive probability, so that in equilibrium all possible p_1 occur with positive probability and there can be no out-of-equilibrium beliefs.

Part 2: Speculators make a positive profit in equilibrium.

Now assume that the market follows its equilibrium strategy. Consider the case where

$p_1 > p_0$, meaning that either $|x_N| < 1$ and $v = v_H$ or simply $x_N \geq 1$.[14] If they follow their equilibrium strategies, the speculators' buy orders will drive the price to $p_0 + \lambda(x_N + 1) > p_0$, and in period 2 all speculators will be able to sell their holdings at $p_2^H = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L$. Their profit is then $p_2^H - p_1$, or $\pi(p_1)v_H + (1 - \pi(p_1))\, v_L - p_0 - \lambda(x_N + 1)$, which can also be written as

$$\frac{F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)}{4 - 2F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - 2F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)} (v_H - v_L) - \lambda\left(x_N + 1\right)$$

Since every speculator knows x_N, the expected profit is

$$E\left[\left.\frac{F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)}{4 - 2F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - 2F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)}\right| x_N\right] (v_H - v_L) - \frac{\hat{\lambda}}{2}\left(x_N + 1\right)$$

Since p_1 is increasing in x_N, $F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right)$ and $F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)$ are also (weakly) increasing in x_N. The expression therefore becomes minimal for $x_N = n$. If at this minimal point it is still non-negative, speculators make a positive expected profit in equilibrium; this is the case if

$$\hat{\lambda} \leq E\left[\left.\frac{F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)}{4 - 2F\left(\max\left\{\frac{p_1-p_0}{\hat{\lambda}} - 1,\, 1\right\}\right) - 2F\left(\frac{p_1-p_0}{\hat{\lambda}} - 1\right)}\right| x_N = n\right] \frac{(v_H - v_L)}{n + 1}$$

This gives a minimum condition for market depth, which is simply given by the spread between high and low value, adjusted for the number of market participants and some adjustment factor that depends on the precise shape of F. If this minimum condition is fulfilled, speculators make an expected profit in equilibrium.

Part 3: No single speculator has an incentive to deviate from his equilibrium strategy.

Part 2 shows that every speculator has, after having observed v and x_N, a non-negative expected profit from following his equilibrium strategy. If his equilibrium action is to buy, then $p_1 - p_0 \geq 0$, and $p_0 - p_1 \geq 0$ if his equilibrium action is to sell. If he were to do nothing instead, his profit would 0, which is not better. If he were to do the opposite, his profit would be non-positive, which is also not an improvement. All speculators hence optimally follow their equilibrium strategies.

Proof of Proposition 3. Let $u_B(\omega^*, \varsigma)$ be the expected utility for a single speculator from buying if he observes ω^* and all other speculators follow the strategy ς. Let $x_S(x_N^*, \varsigma)$ be the order flow from the speculators if x_N takes the value x_N^* and all speculators follow the strategy ς.

[14] An analogous argument applies where $p_1 < p_0$.

Let $u_S(\omega^*, \varsigma)$ be the expected utility from selling after observing ω^* if all others follow the strategy ς. Since the single speculator has no market power, it is $u_B(\omega^*, \varsigma) > 0 \Leftrightarrow u_S(\omega^*, \varsigma) < 0$ and vice versa. Every speculator, taking the strategy ς played by others as given, should then buy if $u_B(\omega^*, \varsigma) > 0$ and sell otherwise.

Now consider the strategy where a speculator buys only if ω is above some threshold k, and call this strategy ς_k. Can there be an equilibrium where all speculators follow ς_{k^*} and where $u_B(k^*, \varsigma_{k^*}) > 0$? No, because then there would be a neighborhood $B_\delta(k^*)$ (with $\delta > 0$) around k^* so that $u_B(k, \varsigma_k) > 0$ if $k \in B_\delta(k^*)$. Then it would not be optimal to follow ς_{k^*} if all others follow ς_{k^*}. A similar argument applies to $u_S(k^*, \varsigma_{k^*})$. Hence we know that in equilibrium, it has to be $u_B(k^*, \varsigma_{k^*}) = u_S(k^*, \varsigma_{k^*}) = 0$.

To find all k^* for which this is true, we need to describe the shape of $u_B(k^*, \varsigma_{k^*})$. A speculator observing ω only knows that the observations of the other speculators are within $[\omega - \varepsilon,\, \omega + \varepsilon]$. x_S depends on how many speculators have a higher observation than ω and how many have a lower observation. Let $\varphi_\omega \in [0, 1]$ be the proportion of speculators that have an observation larger than ω. Then, from the viewpoint of the single speculator who observes ω, φ_ω is uniformly distributed on $[0, 1]$. Then, again from the viewpoint of the single speculators, x_S is uniformly distributed on $[-1,\, 1]$.

Let $\Pi_B(x_N, x_S)$ be the expected profit of a single speculator if noise order flow is x_N and speculator order flow x_S and the speculator has bought. Then

$$u_B(k, \varsigma_k) = \frac{1}{2}\int_{-1}^{1} \Pi_B(k,\, x_S)\, dx_S. \tag{7}$$

$\Pi_B < 0$ if $x_N + x_S < 0$ and $\Pi_B > 0$ if $x_N + x_S > 0$. We can therefore rewrite (7) as

$$u_B(k, \varsigma_k) = \frac{1}{2}\left(\int_{-1}^{-k} \Pi_B(k, x_S)\, dx_S + \int_{-k}^{1} \Pi_B(k, x_S)\, dx_S\right),$$

where the first integral describes an area below the x_S-axis (i.e. sums up only negative values) while the second integral describes an area above the x_S-axis. Due to symmetry in x_N and $\pi(p_1)$, it is $\Pi_B(x_N, x_S) = -\Pi_B(x_N, x_S^{'})$ if $x_N + x_S = -(x_N + x_S^{'})$. Then we can immediately see that $u_B(0, \varsigma_0) = 0$. Now assume that $k < 0$. Then we can disaggregate (7) further:

$$2u_B(k, \varsigma_k) = \int_{-1}^{-1-2k} \Pi_B(k, x_S)\, dx_S + \int_{-1-2k}^{-k} \Pi_B(k, x_S)\, dx_S + \int_{-k}^{1} \Pi_B(k, x_S)\, dx_S$$

Here, the second and the third term add to zero, while the first term is clearly negative.

Therefore, for $k < 0$, the whole expression is negative. A symmetrical argument applies to $u_S(k, \varsigma_k)$.

We can therefore say:

$$u_B(k, \varsigma_k) = \begin{cases} < 0 & \text{if } k < 0 \\ 0 & \text{if } k = 0 \\ > 0 & \text{if } k > 0 \end{cases}$$

Therefore, the only equilibrium is that all speculators follow the strategy ς_0: "Sell if $\omega < 0$ and buy if $\omega > 0$." □

B Which Equilibrium is Pareto-Preferred?

Proposition 4. *If $f(x)$ is falling in $|x|$ (with $x \in \mathbb{N}$), speculators prefer the adjustment to the non-adjustment equilibrium.*

We make use of the following lemma:

Lemma 1. *If $2m > \frac{p_1 - p_0}{\hat{\lambda}}$ it is $\frac{\partial p_2^H(p_1)}{\partial p_1} < 0$ (and hence also $\frac{\partial p_2^L(p_1)}{\partial p_1} > 0$). If $2m \leq \frac{p_1 - p_0}{\hat{\lambda}}$, then $p_2^H = p_2^L = p_0$ and consequentially $\frac{\partial p_2^H(p_1)}{\partial p_1} = \frac{\partial p_2^L(p_1)}{\partial p_1} = 0$.*

Proof. It is $p_2^H(p_1) = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L$, or

$$p_2^H(p_1) = \frac{1 - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)}{2 - F(k_H) - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)} v_H + \frac{1 - F(k_H)}{2 - F(k_H) - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)} v_L.$$

Since $k_H = \max\left\{\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right), m\right\}$, there are two possible cases:

1. $2m > \frac{p_1 - p_0}{\hat{\lambda}}$. Then $k_H = m$ and

$$p_2^H(p_1) = \frac{1 - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)}{2 - F(m) - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)} v_H + \frac{1 - F(m)}{2 - F(m) - F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)} v_L.$$

   As $F\left(\frac{p_1 - p_0}{\hat{\lambda}} - m\right)$ is monotonously growing in p_1, and since $v_H > v_L$, it is then $\frac{\partial p_2^H(p_1)}{\partial p_1} < 0$.

2. $2m \leq \frac{p_1 - p_0}{\hat{\lambda}}$. Then $k_H = \frac{p_1 - p_0}{\hat{\lambda}} - m$ and

$$p_2^H(p_1) = \frac{v_H + v_L}{2} = p_0.$$

□

Proof of Proposition 4. Speculators' expected profit from the efficient equilibrium is the sum of expected profits if $|x_N| < m$ (i.e. if there is no bubble) and $|x_N| \geq m$ (if there is a bubble). More precisely, it is

$$\Pr(|x_N| < m)\left(E\left[p_2^H(p_1)||x_N| < m\right] - p_0 - \frac{\hat{\lambda}}{2}\left(E\left[x_N||x_N| < m\right] + m\right)\right)$$

$$+\Pr(|x_N| = m)\left(E\left[p_2^H(2\lambda m)\right] - p_0 - \frac{\hat{\lambda}}{2}(2m)\right) \tag{8}$$

$$+\Pr(|x_N| > m)\left(E\left[p_2^H(p_1)|x_N > m\right] - p_0 - \frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] + m\right)\right)$$

(Note that, because of symmetry, we can restrict ourselves to the expected prices if $p_1 > p_0$.) All three summands are clearly positive, as we can see from lemma 1 and the proof of proposition 1.

In the inefficient equilibrium, expected profit for any speculator is positive only if $|x_N| > m$, so that overall expected profit from the inefficient equilibrium is

$$\Pr(|x_N| > m)\frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] - m\right).$$

If the expression "(Expected profit from efficient equilibrium)−(Expected profit from inefficient equilibrium)" is positive, speculators prefer the efficient equilibrium. We can write this expression as the sum of some positive terms and the term

$$\Pr(|x_N| > m)\left(E\left[p_2^H(p_1)|x_N > m\right] - p_0 - \frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] + m\right) - \frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] - m\right)\right). \tag{9}$$

From the proof of proposition 1 we know that $E\left[p_2^H(\lambda(n+m))\right] - p_0 - \frac{\hat{\lambda}}{2}(n+m) > 0$. From lemma 1, it follows that then also $E\left[p_2^H(\lambda(x_N+m))|x_N > m\right] > p_0 + \frac{\hat{\lambda}}{2}(n+m)$. That means that if

$$\frac{\hat{\lambda}}{2}(n+m) - \frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] + m\right) - \frac{\hat{\lambda}}{2}\left(E\left[x_N|x_N > m\right] - m\right) \tag{10}$$

is positive, then expression (9) is also positive. (10) simplifies to $n + m - 2E\left[x_N|x_N > m\right]$, which is positive if $\frac{n+m}{2} > E\left[x_N|x_N > m\right]$. If $f(x)$ is falling in $|x|$, that is the case. □

It should be noted that this is a sufficient, but not a necessary condition: The difference between expected payoffs from the efficient and the inefficient equilibrium can well be positive even if $\frac{n+m}{2} < E\left[x_N|x_N > m\right]$. But it can be shown that the efficient equilibrium is not always preferred: If f is not falling in its argument, it is possible that speculators actually prefer the inefficient equilibrium. Intuitively, that is the case if f has a lot of mass towards n and $-n$, so that large bubbles (which are profitable for rational speculators in the inefficient equilibrium) become very likely. In the efficient equilibrium, the market adjusts $\pi(p_1)$ accordingly, and speculators' expected profit margins in the efficient equilibrium (which is now not very efficient) become very low. In the inefficient equilibrium, on the other hand, speculators could now make large expected gains, since their profit is higher the further noise traders drive p_1 away from p_0.

Corollary. *There are distributions of x_N so that the efficient equilibrium exists, but speculators ex ante prefer the inefficient equilibrium.*

Proof. Consider the the case where $\Pr(x_N = 1) = \Pr(x_N = -1) = \varepsilon$ and $\Pr(x_N = n) = \Pr(x_N = -n) = \frac{1}{2} - \varepsilon$. Then the expected payoff in the inefficient equilibrium is $(1-2\varepsilon)\frac{\hat{\lambda}}{2}(n-m)$, while the expected payoff from the efficient equilibrium is

$$E\left[\varepsilon p_2^H\left(p_0 + \lambda(m + x_N)\right)\middle| x_N = 1\right] + E\left[\varepsilon p_2^H\left(p_0 + \lambda(m + x_N)\right)\middle| x_N = -1\right]$$

$$+E\left[(1-2\varepsilon)\, p_2^H\left(p_0 + \lambda(m + x_N)\right)\middle| x_N = n\right] - \frac{\hat{\lambda}}{2}m - (1-2\varepsilon)\frac{\hat{\lambda}}{2}n - p_0.$$

Let $D_i = E\left[p_2^H\left(p_0 + \lambda(m + x_N)\right)\middle| x_N = i\right] - p_0$. Then the difference between profits from the efficient and inefficient equilibrium is

$$\varepsilon D_1 + \varepsilon D_{-1} + (1-2\varepsilon)D_n - \hat{\lambda}\left(\varepsilon m + (1-2\varepsilon)n\right). \tag{11}$$

If we take the maximal $\hat{\lambda}$ such that the efficient equilibrium still exists,[15] we have $\hat{\lambda} = 2\frac{D_n}{m+n}$, and (11) becomes

$$\varepsilon D_1 + \varepsilon D_{-1} + \frac{(1-4\varepsilon)m - (1-2\varepsilon)n}{m+n} D_n.$$

For this always to be positive, it would have to be

$$\frac{D_1 + D_{-1}}{2} / D_n > \frac{4\varepsilon m - 2\varepsilon n - m + n}{2\varepsilon(m+n)}.$$

Intuitively, this means that as ε gets arbitrarily small, the prices that result in period 1 from $x_N = 1$ and $x_N = -1$ would have to become infinitely more informative than the prices that result from $x_N = n$ and $x_N = -n$. But a price p_1 that results from $x_N = n$ lies within the price range $\left[p_0 + \hat{\lambda}(-m-1),\, p_0 + \hat{\lambda}(m+1)\right]$ with constant probability $\frac{m+1}{m+n}$ because of the price formation process through noisy λ. Therefore, the prices resulting from $x_N = 1$ and $x_N = -1$ can never be infinitely more informative than the prices resulting from $x_N = n$. Therefore, there

[15] For very small $\hat{\lambda}$, speculators always prefer the efficient equilibrium.

exists a distribution for x_N so that for a large enough $\hat{\lambda}$ speculators prefer the inefficient to the efficient equilibrium. $\square$

These conditions on the shape of f might seem rather abstract, but they have an intuitive interpretation. f is falling in distance from 0 if the correlation between noise traders' decisions is sufficiently small (they might make their decisions independently, or their actions might even be negatively correlated). In these cases, speculators will always prefer the efficient equilibrium. But high correlation between the decisions of the noise traders means nothing else than strong herding. If noise traders are sufficiently prone to strong herding, all rational market participants weakly prefer an equilibrium in which no information is transmitted to a partially revealing equilibrium.

We should not forget the social implications of the two equilibria beyond the scope of this model: Society as a whole will presumably at all times prefer if information is transferred to the market price, rather than having speculators that know something about the asset sit on that information without sharing it.

C The Speculator's Game

Usually, the efficient equilibrium is preferred by speculators to the inefficient equilibrium. (The market in period 2 never makes non-zero profit and is therefore indifferent between all equilibria.) Intuitively, this is the case because speculators only profit in the inefficient equilibrium if $|x_N| \geq 1$, but always (except perhaps if $x_N = n$) in the efficient equilibrium. Therefore, if f (the distribution of x_N) is falling with distance from its mode 0, making a profit in the inefficient equilibrium is sufficiently unlikely to make the efficient equilibrium more profitable in expected terms regardless of the precise shape of f. If the correlation between the behavior of noise traders is sufficiently low (i.e. negative or zero), that is always the case. But if the correlation between noise trader's action is positive and large enough, however, the inefficient equilibrium can actually be pareto-preferred. See appendix B for proofs and a brief discussion of these results.

It seems discouraging enough that rational traders can under some circumstances prefer to keep their information for themselves. But if we say that the correlation between noise traders' orders is low enough so that the efficient equilibrium is preferred—will it at least be stable? In the efficient equilibrium, the speculators must often "overcome" the noise traders to submit information. This requires some coordination, and most of all trust in the other speculators.

Let us consider the situation where the asset has high value ($v = v_H$) while the noise order flow is wrong but sufficiently small ($x_N \in [-m+1, -1]$). In the efficient equilibrium, speculators will buy and drive the price p_1 above p_0, after which they will earn a positive profit when selling the asset in period 2. But consider the situation of a single speculator: He knows that he will make a profit by buying only if enough other speculators do the same. If not enough others join him, he would do better by *selling*. In any case, he can avoid any risk by abstaining from either buying or selling.

If we assume that the market "listens" to p_1 and take its strategy as given, we can exemplify the single speculator's dilemma in a simplified partial game with 3 speculators, among whom it is common knowledge that $x_N = -2$ and $v = v_H$. They each have to choose whether to buy (B, their action in the efficient equilibrium), do nothing (N) or sell (S). The resulting game is shown in figure 6. To simplify matters, positive payoffs are marked as (+), and negative payoffs simply as (-). While positive (or negative) payoffs might differ from one another, these differences are irrelevant if we only want to consider the best responses.

	B	N	S
B	+,+,+	0,0,0	-,-,+
N	0,0,0	-,0,0	-,0,+
S	-,+,-	-,+,0	-,+,+

B

	B	N	S
B	0,0,0	0,-,0	0,-,+
N	0,0,-	0,0,0	0,0,+
S	0,+,-	0,+,0	0,+,+

N

	B	N	S
B	+,-,-	+,-,0	+,-,+
N	+,0,-	+,0,0	+,0,+
S	+,+,-	+,+,0	+,+,+

S

Figure 6: Simplified game involving three speculators, if $x_N = -2$ and $v = v_H$. Each speculator chooses whether to buy (B), do nothing (N) or sell (S). Speculator 1 chooses matrix, 2 chooses row and 3 column. Differences among positive and negative payoffs are irrelevant for best responses. All positive payoffs constitute best responses.

(B, B, B), which is the equilibrium strategy profile for the efficient equilibrium, is also an equilibrium in this partial game. But it is unstable: As soon as one of the speculators changes his strategy, all of the others have an incentive to also change their strategies. In fact, B is weakly dominated by S for all speculators, and the equilibrium (B, B, B) is trembling-hand imperfect.

In return, this partial game has a new equilibrium (S, S, S) where all three speculators do the opposite of their efficient-equilibrium action. This is extremely stable: A speculator has no incentive to change his strategy away from S even if he thinks that the other two will.

If all three speculators choose S, collapse of the efficient equilibrium will of course be the consequence, because p_1 is now actually tricking the market into believing that $v = v_L$.

If we concentrate on the problem of one single speculator and take not only the strategy of the market, but also the actions of all other speculators as given, we arrive at an overview over the speculator's choices shown in table 1. The single speculator has to choose whether to buy (B), do nothing (N) or sell (S), but now we take the net order flow from other speculators $x_S^{'}$ as given. We also assume (more generally than before) that $v = v_H$ and $x_N < 0$.

	$x_S^{'} > -x_N + 1$	$x_S^{'} = -x_N + 1$	$x_S^{'} = -x_N$	$x_S^{'} = x_N - 1$	$x_S^{'} < -x_N - 1$
B	+	+	+	0	-
N	0	0	0	0	0
S	-	0	+	+	+

Table 1: One speculator's payoff from buying (B), doing nothing (N) or selling (S) depending on the actions of other speculators. $x_S^{'}$ is the net order flow from other speculators. Positive (+) and negative (-) payoffs may differ, but these differences are not relevant if we consider best responses. If $x_S^{'} = -x_N$, payoffs from buying or selling are identical.

It is optimal for the speculator to buy in the first three cases—if either all other speculators together have already neutralized the noise or even pushed p_1 above p_0. In the three latter cases, it is optimal for the speculator to do the opposite from his equilibrium action—if he can push p_1 below p_0 by doing so, or if the other speculators have not pushed p_1 above p_0 in the first place.

The strength of these incentives changes with the difference of m and x_N. If $m = x_N + 1$ (as in the three-player example above), speculators would correct the price in the efficient equilibrium, but every single speculator is indifferent between following his equilibrium strategy and doing exactly the opposite. And if just one speculator changes his strategy, it is not longer optimal for any other speculator to buy.

The situation of the speculators has similarities to the "stag hunt" game after Rousseau, which has become a staple in teaching game theory. If all speculators work together, they can "bag the stag"—here: transmit information about the value, and profit from it. If sufficiently many of them drop out, information transmission is no longer profitable. The smaller the difference between m and $|x_N|$ (i.e. the more wrong the noise is), the fewer speculators need to deviate to make deviation optimal for all speculators.

The most tenuous feature of the stag hunt game is present here, too: If a speculator believes that others are not following their equilibrium strategies, it becomes optimal for him to deviate—thus making it more likely that not enough speculators are choosing the equilibrium action. And a speculator who believes that others think that he will deviate will also believe that these others will therefore likely deviate themselves, making it more attractive to deviate for himself. We should therefore investigate the influence of higher-order beliefs on equilibrium selection.

D Extension: An Informed Market

The question that the model in this paper tries to answer is: Will short-term interested speculators, if they have information about the fundamental value of an asset, still trade on that information if other market participants do not have that information? Therefore, a central assumption of the model is that the speculators are informed about the true value of the asset, while the market only knows the unconditional prior. But we can also ask what happens to the main predictions of the model if we relax or remove this assumption.

Consider the following modification: Instead of being completely uninformed, the whole market will learn v with probability γ at the beginning of period 2.

Now, if the speculators know whether the market will be informed in period 2, we have two trivial cases: One, in which the market will receive the information, and the speculators therefore all buy in period 1 if $v = v_H$ and sell otherwise, and one case in which the market will not receive information and everything is as before. But what if the speculators do not know whether the market will receive information?

Then the expectation of p_2 for a speculator is no longer p_0, but $\gamma v + (1-\gamma)p_0$. A speculator who observes v_H now expects a second-period price of $p_0 + \gamma\left(\frac{v_H - v_L}{2}\right) > p_0$. Therefore, if $x_N < 0$ it may no longer be an equilibrium that $-x_N$ speculators buy the asset such that $x = 0$ and $p_1 = p_0$. If $p_0 + \gamma\left(\frac{v_H - v_L}{2}\right) \geq p_0 + \frac{\hat{\lambda}}{2}$ (where the latter is the expected p_1 if $x = 1$), it makes sense for an additional speculators to buy the asset (if possible). This is equivalent to $\gamma \geq \frac{\hat{\lambda}}{v_H - v_L}$. (If also $p_0 + \gamma\left(\frac{v_H - v_L}{2}\right) \geq p_0 + \hat{\lambda}$, it is then also optimal for another speculator to buy and so on.)

But by trading in this way, the speculators actually reveal information about v, since they would not be buying beyond $x = 0$ if $v = v_L$ and the expected p_2 was $\gamma v_L + (1-\gamma)p_0 = p_0 - \gamma\left(\frac{v_H - v_L}{2}\right)$. It is then also no longer optimal for the market in period 2 to assume that p_1 is uninformative, and to instead condition p_2 on p_1.

The conclusion of this exercise is therefore: If there is some probability that the full market will receive the same information as the speculators before the speculators have to liquidate their holdings, and if the speculators cannot know beforehand whether the market will learn this information or not, and if the probability that the market will learn the information is sufficiently high (i.e. $\gamma \geq \frac{\hat{\lambda}}{v_H - v_L}$), then the inefficient equilibrium will not exist.

What about the efficient equilibrium? If $v = v_H$ and $p_1 > p_0$ (or $v = v_L$ and $p_1 < p_0$), the speculators' expected profits actually rise as γ gets larger, and nothing changes in the efficient equilibrium. The change is in the fact that the "bubbles" in the efficient equilibrium—i.e. where speculators trade against their own information because $|x_N| \geq m$—become less profitable for the speculators, because they lose money with some probability. Let $E_B = E\left[p_2^H(p_0 + \lambda(x_N + m))\right]$ and $E_S = E\left[p_2^H\left(p_0 + \lambda(x_N + m - 2)\right)\right]$, and consider the expected payoff for the marginal speculators of buying when v_L and $x_N \geq m$ and the market learns v with probability γ:

$$\gamma v_L + (1-\gamma)E_B - p_0 - \frac{\hat{\lambda}}{2}(x_N + m)$$

The expected payoff of selling for the same marginal speculator is

$$-\gamma v_L - (1-\gamma)E_S + p_0 + \frac{\hat{\lambda}}{2}(x_N + m - 2).$$

Therefore, the "last" speculator decides to sell instead of buying if

$$\begin{aligned} \gamma &\geq \frac{E_B + E_S - 2p_0 - \hat{\lambda}(m + x_N - 1)}{E_B + E_S - 2v_L} \\ &\geq 1 - \frac{2p_0 + \hat{\lambda}(m + x_N - 1) - 2v_L}{E_B + E_S - 2v_L} \end{aligned}$$

The efficient equilibrium in the form described in this paper continues to exist if this condition on γ is not fulfilled. For $\hat{\lambda}$ close to the maximum given by condition 6 the right-hand side term is very small, because $p_0 + \frac{\hat{\lambda}}{2}(m + x_N - 1)$ is very close to $\frac{E_B+E_S}{2}$. For $\hat{\lambda} \to 0$ it converges against 1.

E A Discrete Model where Speculators have Market Power

The model can also be written with a finite number of speculators and noise traders, such that signle speculators actually have market power and can influence the price. While this makes some of the expressions less tractable and slightly changes the proofs, the main theorems remain intact and the two equilibria still exist.

Proposition 5. *(Efficient equilibrium) It is an equilibrium if every speculator follows the strategy "If $x_N \leq -1$, sell and if $x_N \geq 1$ buy. If $|x_N| < 1$, buy if $v = v_H$ and sell if $v = v_L$." and the market sets*

$$p_2 = p_2^H(p_1) = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L \quad \text{if} \quad p_1 > p_0 \tag{12}$$

$$p_2 = p_2^L(p_1) = (1 - \pi(p_1))\, v_H + \pi(p_1)v_L \quad \text{if} \quad p_1 < p_0 \tag{13}$$

$$p_2 = p_0 \quad \text{if} \quad p_1 = p_0, \tag{14}$$

where

$$\pi(p_1) = \begin{cases} \frac{1-F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m \right\rfloor\right)}{2-F(k_H)-F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m \right\rfloor\right)} & \text{if } p_1 > p_0 \\ \\ \frac{1-F\left(\left\lceil \frac{p_1-p_0}{\hat{\lambda}} + m \right\rceil\right)}{2-F(k_L)-F\left(\left\lceil \frac{p_1-p_0}{\hat{\lambda}} + m \right\rceil\right)} & \text{if } p_1 < p_0 \end{cases}$$

where $\pi(p_1)$ is the market's belief that $v = v_H$ if $p_1 > p_0$ or that $v = v_L$ if $p_1 < p_0$, respectively, with $k_H = \max\left\{\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m \right\rfloor, m-1\right\}$ and $k_L = \min\left\{\left\lceil \frac{p_1-p_0}{\hat{\lambda}} + m \right\rceil, -m+1\right\}$, if and only if

$$\hat{\lambda} \leq E\left[\frac{F(k_H) - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m \right\rfloor\right)}{2 - F\left(k_H\right) - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m \right\rfloor\right)} \middle| x_N = n\right] \frac{v_H - v_L}{m+n}. \tag{15}$$

Proof (similar to the continuous case). **Part 1: The market has no incentive to deviate (and $\pi(p_1)$ is the correct belief).**

Assume that the speculators follow their equilibrium strategies and consider the case where $p_1 > p_0$. The market can then, from observing p_1, draw conclusions about v. Let $\pi(p_1)$ be the conditional probability that $v = v_H$ after observing a certain p_1, $\Pr(v_H|p_1)$.

It is

$$\pi(p_1) = \Pr(v_H|p_1) = \frac{\Pr(p_1 \cap v_H)}{\Pr(p_1)} = \frac{\Pr(p_1 \cap v_H \cap |x_N| < m) + \Pr\left(p_1 \cap v_H \cap x_N \geq m\right)}{\Pr\left(p_1 \cap |x_N| < m\right) + \Pr\left(p_1 \cap x_N \geq m\right)}$$

since $\Pr(p_1 \cap x_N \leq -m) = 0$.

If g is the probability density function of λ, we can express this as

$$\pi(p_1) = \frac{\frac{1}{2}\sum_{y=-m+1}^{m-1} f(y)g\left(\frac{p_1-p_0}{y+m}\right) + \frac{1}{2}\sum_{y=m}^{n} f(y)g\left(\frac{p_1-p_0}{y+m}\right)}{\frac{1}{2}\sum_{y=-m+1}^{m-1} f(y)g\left(\frac{p_1-p_0}{y+m}\right) + \sum_{y=m}^{n} f(y)g\left(\frac{p_1-p_0}{y+m}\right)}.$$

The product in all the sums, $f(y)g\left(\frac{p_1-p_0}{y+m}\right)$, gives the probability that $x_N = y$ and $\lambda = \frac{p_1-p_0}{y+m}$, in which case the parameters would lead to the given p_1 if speculators always bought in period 1. The first sum in the numerator is hence the overall probability that p_1 would be observed as a result of some $x_N \in [-m+1, m-1]$ if speculators always bought the asset. Since, if $x_N \in [-m+1, m-1]$, speculators buy the asset only if $v = v_H$, this probability has to be multiplied by $\frac{1}{2}$ to give the probability $\Pr(p_1 \cap v_H \cap |x_N| < m)$. The second sum in the numerator gives the probability that p_1 would be observed as the result of some $x_N \geq m$. Since $v = v_H$ in only half of these cases, we again need to multiply with $\frac{1}{2}$ (albeit for different reasons) to get the unconditional probability that p_1 would happen as the result of some $x_N > m$ *and* that also $v = v_H$. In the numerator, therefore, we have the overall probability that a given p_1 is observed and is informative.

In the denominator, we then have the overall probability that a given p_1 is observed. This is given by the expression from the numerator, only that now *all* cases in which $x_N > m$ are considered (since they all lead to $p_1 > p_0$), whereas only half of them are informative. The fraction therefore gives the ratio between the number of cases in which p_1 is observed and it is $v = v_H$ and the overall number of cases in which p_1 is observed. This is the conditional probability $\Pr(v_H|p_1)$.

We can simplify the expression: Since λ is uniformly distributed on the interval $\left(0, \hat{\lambda}\right)$, $g\left(\frac{p_1-p_0}{y+m}\right) = \frac{1}{\hat{\lambda}}$ if $0 < \frac{p_1-p_0}{y+m} < \hat{\lambda}$ and 0 otherwise. For any $p_1 > 0$, it is $0 < \frac{p_1-p_0}{y+m}$, but $g\left(\frac{p_1-p_0}{y+m}\right)$ is nonzero only for $y > \frac{p_1-p_0}{\hat{\lambda}} - m$. We can write

$$\begin{aligned}
\pi(p_1) &= \frac{\sum\limits_{y=\left\lceil \frac{p_1-p_0}{\hat{\lambda}}-m\right\rceil}^{k_H} f(y) + \sum\limits_{y=k_H+1}^{n} f(y)}{\sum\limits_{y=\left\lceil \frac{p_1-p_0}{\hat{\lambda}}-m\right\rceil}^{k_H} f(y) + 2\sum\limits_{y=k_H+1}^{n} f(y)} \\
&= \frac{1 - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor\right)}{2 - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor\right) - F\left(k_H\right)}
\end{aligned}$$

where $k_H = \max\left\{\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor, m-1\right\}$. Therefore, given the speculators' strategies, $\pi(p_1)$ gives the correct beliefs in equilibrium.

$\Pr(v_L|p_1)$ is the complementary probability $1 - \pi(p_1)$, so that the expected value of v given p_1 is $E\left[v|p_1\right] = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L$. A similar argument applies to the case where $p_1 < p_0$. If $p_1 = p_0$, the price contains no information and p_2 should be set equal to the prior.

p_1 is between $p_0 - \hat{\lambda}(m+n)$ and $p_0 + \hat{\lambda}(m+n)$. For $x_N \in \{-n, n\}$, all possible p_1 occur with positive probability, so that in equilibrium (where $x_N \in [-n, n]$) all possible p_1 occur with positive probability and there can be no out-of-equilibrium beliefs.

Part 2: Speculators make a positive profit in equilibrium.

Now assume that the market follows its equilibrium strategy. Consider the case where $p_1 > p_0$, meaning that either $|x_N| < m$ and $v = v_H$ or simply $x_N \geq m$. If they follow their equilibrium strategies, the speculators' buy orders will drive the price to $p_0 + \lambda(m + x_N) > p_0$, and in period 2 all speculators will be able to sell their holdings at $p_2^H = \pi(p_1)v_H + (1 - \pi(p_1))\, v_L$. Their profit is then $p_2^H - p_1$, or $\pi(p_1)v_H + (1 - \pi(p_1))\, v_L - p_0 - \lambda(m + x_N)$, which can also be written as

$$\left[\frac{1 - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor\right)}{2 - F\left(k_H\right) - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor\right)} - \frac{1}{2}\right] v_H + \left[\frac{1 - F\left(k_H\right)}{2 - F\left(k_H\right) - F\left(\left\lfloor \frac{p_1-p_0}{\hat{\lambda}} - m\right\rfloor\right)} - \frac{1}{2}\right] v_L - \lambda(m + x_N) \tag{16}$$

$$= \frac{F(k_H) - F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)}{4 - 2F\left(k_H\right) - 2F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)}(v_H - v_L) - \lambda(m + x_N)$$

x_N is known to the speculators. Then we can write expression 16 in expected terms (given x_N):

$$E\left[\left. \frac{F(k_H) - F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)}{4 - 2F\left(k_H\right) - 2F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)} \right| x_N \right] (v_H - v_L) - \frac{\hat{\lambda}}{2}(m + x_N).$$

Since p_1 is monotonically increasing in x_N, and therefore $F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)$ and $F(k_H - 1)$ are weakly increasing in x_N, the whole expression becomes minimal for $x_N = n$, where it is

$$E\left[\left. \frac{F(k_H) - F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)}{4 - 2F\left(k_H\right) - 2F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)} \right| x_N = n \right] (v_H - v_L) - \frac{\hat{\lambda}}{2}(m + n).$$

If this is positive, then speculators will make an expected profit by following their equilibrium strategies for all x_N (the case where x_N is negative is analogous and leads to the same result). We can reformulate the condition as

$$\hat{\lambda} \leq E\left[\left. \frac{F(k_H) - F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)}{2 - F\left(k_H\right) - F\left(\left\lfloor \frac{p_1 - p_0}{\hat{\lambda}} - m \right\rfloor\right)} \right| x_N = n \right] \frac{v_H - v_L}{m + n}$$

which is simply the spread between high and low value, adjusted for the number of market participants and some adjustment factor that depends on the precise shape of f.

Part 3: No single speculator has an incentive to deviate from his equilibrium strategy.

As speculators always make a profit in equilibrium, it would not be profitable for any speculator to deviate by not trading at all. But what if a speculator decided to sell if his equilibrium action would be to buy? We have to distinguish three cases (note that “buy” would never be an equilibrium action if $x_N \leq -m$):

1. $x_N = -(m - 1)$. In this case it is $x = 1$ in equilibrium, and if a single speculator decided to sell instead of buying, x would be -1. Since $p_2(p_1)$ is point-symmetric around (p_0, p_0) (i.e. $p_2(p_1) - p_0 = p_0 - p_2(p_0 - (p_1 - p_0))$ because of the symmetry assumption on f), the speculator who sold would gain just as much in expectation as he would have by buying. Since he is thus indifferent, there is no incentive to deviate from equilibrium strategies.

2. $x_N = -(m - 2)$. Then $x = 2$ in equilibrium, but if a single speculator sold instead of buying, the resulting net order flow would be 0, so that $p_1 = p_0$. Then it would also be $p_2 = p_0$, so that the speculator would make no gain at all by selling, whereas he could have made a positive profit by buying.

3. $x_N > -(m-2)$. Then $x > 2$ in equilibrium, and a single speculator can only lower x to some slightly lower, but still positive number. Then $p_2 = p_2^H(p_1) > p_1$, so that the speculator would actually make a loss by selling in period 1.

We can therefore conclude that no speculator has an incentive to deviate from his equilibrium strategy if $p_1 > p_0$. A similar argument applies where $p_1 < p_0$ (i.e. if speculators bought instead of selling). □

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