Network formation with value heterogeneity: sorting, adverse effects and structure of networks*
Preliminary draft†

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Abstract

We investigate formation of economic and social networks where agents may form or cut ties. Our novelty is combining a setup with agents having types that may be complementary when forming links with network externalities from indirect connections. We provide sufficient conditions in various settings for sorting in friendships, i.e. agents tend to partner with similar agents. We show that sorting may be stable yet inefficient despite otherwise obeying the conditions for sorting of Becker (1973) - inefficiency stems from the lack of internalizing the benefits that other agents receive from connecting across types. Another novel feature is agents with higher value are more central in networks; a side effect is sorting by degree centrality under certain conditions. Finally we illustrate the limits to patterns of sorting and centrality.

Keywords: assortative matching; assortativity; complementarity; cooperative games; network formation; network externalities; one-sided matching.

JEL classification: C71, C78, D61, D62, D85.

1 Introduction

A classic topic for research on social networks is who connects with whom. The pattern known as *sorting* or homophily where people associate more with others who are similar to themselves has been showed to be ubiquitous for various characteristics e.g. socioeconomic status and personal interests.[1] Economic research has contributed to the understanding of sorting by demonstrating sufficient and necessary conditions for sorting in marriage-labor markets and into groups. However, no research has reconciled Becker (1973)'s framework on sorting with the literature on network formation under externalities (friends of friends impact own utility) as pioneered by Jackson and Wolinsky (1996). Our investigation yields new insights on network formation by rational agents: when is sorting stable and optimal under network externalities, and; which other patterns arise in these stable networks.

New insights on sorting patterns can be employed in various institutions such as schools or labour market. These institutions decide either explicitly or implicitly which individuals get to interact with one another and thus plays a direct role in what social networks are formed.

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[1]See the meta-study McPherson, Smith-Lovin, and Cook (2001) for an overview.

Our framework explores a setting with agents choosing partners under three core assumptions: (i) Agents are heterogeneous in type for creating value in partnerships (this is in line with research on peer effects).[2] Type may refer to productive and non-productive capabilities such as skill, social aptitude or interests and ethnicity. (ii) Possible externalities from friends of friends as in the 'connections model' of Jackson and Wolinsky (1996) which captures spillovers of ideas and favors. (iii) Agents choose a limited number of partners reflecting constraints of time and effort.[3] Note our framework could also model corporations or educational institutions with bilateral partnerships.

In this setting we analyze the structure of networks for robustness in the following sense: no agents can form links and be better off than the current network structure.

Although the literature on formation of networks under externalities is vast the share that investigates heterogeneous agents has been limited to the following. Jackson and Wolinsky (1996) pioneered in their investigation of the tension between stability and efficiency in networks. One class of research on builds on the one-way link formation pioneered by Bala and Goyal (2000) where Goeree, Riedl, and Ule (2009) explores heterogeneity both value and cost in two examples. Note one-way links does not require mutual acceptance which seems appropriate for social networks. Another class of research examines computational feasibility in two settings: finding stable networks under two-way link formation without transfers where externalities arises from caring about others' wellbeing, e.g. Anshelevich, Bhardwaj, and Hoefer (2013); formation of groups (not networks) where structure of one groups affects the other e.g. Michalak, Rahwan, Sroka, Dowell, Wooldridge, McBurney, and Jennings (2009) and Rahwan, Michalak, Wooldridge, and Jennings (2012). A major shortcoming of the latter class is a lack investigation into deeper economic issues and structure.

Below we list our contributions to the literature on network formation and assortative matching. Some of our results are derived when only evaluating links formed between a pair of agents which corresponds to pairwise (Nash) equilibrium. Other results require that any coalition of agents can form links between them which corresponds to strong (Nash) equilibrium.[4]

i. Conditions for stable networks to exhibit sorting in type; see Theorem 1 and Proposition 1. Note sorting is only unique outcome under externalities if population of agents is large.

ii. Exhibit conditions for when a network where types separate into disjoint peer groups with no interconnections is stable but inefficient, see Theorem 2.

iii. Show conditions when agents with high types are more central in stable networks; see Theorem 3 and Proposition 4. Without externalities sorting in degree emerges, see Theorem 4.

iv. Finally we demonstrate limits to patterns above as seen in networks where an agent of lowest type may be excessively central. This network is inefficient while entailing failure of sorting and monotonic centrality. See Example 2 and Corollary 1.

The literature The following paragraphs compare our results with the most relevant work. As there is a large body of literature the review is limited to only the most similar and noteworthy.

The work on sorting (point i.) has a strong tradition for two-sided matching setting such as labor and dating markets starting with Becker (1973). However, the research on one-sided sorting been limited to formation of clubs (not networks) under various technologies cf. Farrell and Scotchmer

[2]Some examples of empirical papers finding this include Zimmer and Toma (2000), Sacerdote (2001), Falk and Ichino (2006). Note that empirical estimates should be cautiously interpreted as the research on peer effects is still in an infant stage cf. Manski (1993), Shalizi and Thomas (2011), Angrist (2013).

[3]Limited partners is also consistent with empirical research; Ugander, Karrer, Backstrom, and Marlow (2011) shows this for the entire Facebook network and Miritello, Lara, and Moro (2013) in phone calls for millions of people.

[4]To the best of the author's knowledge this is the first demonstration of the novel features.

(1988), Kremer (1993), Durlauf and Seshadri (2003), Legros and Newman (2002), Pycia (2012), Baccara and Yariv (2013). None of these consider sorting under externalities.

Research on suboptimal networks under externalities was pioneered by Katz and Shapiro (1985) and Farrell and Saloner (1986) for goods markets and shown generally in (one-sided) network formation by Jackson and Wolinsky (1996).[5] Our contribution for inefficient networks is not to provide any new generic insights on suboptimal network formation; instead we show two new classes of inefficient networks exist: one is over-sorting (point ii.), see paragraph below; the other is when a low value agent is excessively central see last paragraph (point iv.).

The pattern of over-sorting (point ii.) is quite intuitive - when there are spillovers from connecting then agents may not carry the full incentive for connecting the groups. Models of knowledge transfer have looked at optimal network design e.g. Cowan, Jonard, and Zimmermann (2007) which consider strategic link formation in a dynamic setting with type complementarity but ignore externalities; thus the model is also a one-sided sorting models without transfers but it allows for screening through agents past links. The idea of knowledge transfer is also used in the friend suggesting algorithm by Yu, Wang, Bu, Wang, Wu, and Chen (2015) which not only recommend friends who are likely to be good matches but selects friends that are likely to increase knowledge transfer most.

Our results on centrality (point iii.) has related work within economics and network science. The only relevant economic research related to the relationship between type and centrality is Goeree, Riedl, and Ule (2009). They show in two examples a star network with a high value agent in center is efficient and Nash stable. Note their analysis is under one-way formation links and lacks a unique prediction.[6] In their experiments the efficient network emerges more often and the frequency increases with repetitions. For sorting in degree the most related work is Currarini, Jackson, and Pin (2009) who find sorting in degree under homophily but unlike our setup their sorting is driven by distribution of types - this condition is not necessary in our setup. Other models from statistical physics also predict sorting in degree but have no strategic considerations - the most prominent is preferential attachment see Barabasi and Albert (1999).[7] Note that an analogy to our pattern of degree sorting is found in Kremer (1993) and Farrell and Scotchmer (1988) where more productive groups can sustain a larger scale of organizations.

Our results on the limits of sorting (point iv.) indicates that an agent can amass so much network capital that the agent becomes the most central despite not being the best candidate. In these networks there is a failure of sorting and monotonic centrality (point i and iii). The work on network externalities in two-sided settings by Katz and Shapiro (1985) and Farrell and Saloner (1986) find an analogue pattern. In their setups the source is a worse producer supplying the consumers cannot coordinate on choosing the better producer and get "locked-in".

Paper organization The rest of the paper is structured as follows: Section 2 introduces the model; Section 3 investigates sorting and over-sorting; Section 4 analyzes the network structure implied by the model; Section 5 rounds up the analysis by outlining the limits to sorting and network structure, and; Section 6 concludes in a discussion of assumptions.

[5]Note our concept of pairwise stability differs from Jackson and Wolinsky (1996) - see Section 2 (the model).

[6]Other star networks are also stable. Moreover, star networks where periphery agents pay for links may be strongly stable (for some parameters) due to absense in transfers of utility.

[7]A drawback of preferential attachment is that including a parameter of attractiveness similar to value heterogeneity then there is negative degree assortativity, cf. Pastor-Satorras, Vazquez, and Vespignani (2001). In a related model Konig, Tessone, and Zenou (2010) shows through simulation that when allowing for preferential attachment with local search (through friends) and capacity constraints (similar to our framework) then degree assortativity also emerges.

2 Link formation with heterogeneous peers

Let N constitute a set of n agents. Each agent $i \in N$ is endowed a fixed measure of *type*, $x_i \in X$ where X is the set of realized levels of types for agents in N. Let $\bar{x} = \max X$ and $\underline{x} = \min X$. Let the agents' type be sorted descending and captured by $\mathcal{X} = (x_1, x_2, ..., x_n)$ and denote $\bar{x} = \max X, \underline{x} = \min X$.

A *coalition* of agents $t \subseteq N$ is an element in T which is the superset of N excluding the empty set. For a given group t define $\mathcal{X}(t)$ as the vector of types ordered descending for agents in t.

Linking and networks Two agents $i, j \in N$ may *link* if mutually accepted; a link may be broken by both agents. A link between i and j is denoted $ij \in \mu$ where the set μ consists of links and is called a *network*. The set of all networks is denoted $M = \{\mu | \, \mu \subseteq \mu^c\}$ where μ^c is the *complete* network (all agents are linked).

Define the *neighborhood* of i in network μ as the subset agents linked to i - this is denoted $\nu_i(\mu) = \{j \in N : ij \in \mu\}$. The count of neighbors is the first measure of centrality called *degree* and denoted $k_i(\cdot)$ for i. For t define $\mathcal{K}(t)$ as the vector of degrees for agents in t ordered descending.

A *walk* from agent i_1 to agent i_l is a subset of links $\{i_1 i_2, i_2 i_3, ..., i_{l-1} i_l\} \subseteq \mu$ - the length of the walk is the number of links in the subset. A *path* is a walk where no agent is reached more than once in the walk. The *distance* between two agents i, j in the network is the length of the shortest path - this is denoted p_{ij}. When no path exist then distance is infinite.

Utility The utility accruing to agent i is denoted u_i. An agent's utility equal benefits subtracted costs which expressed mathematically as $u_i = b_i - c_i$. The aggregate utility is denoted $U(\cdot)$.

Two approaches are used for modeling costs. First, imposing a *degree quota*, κ, which is an upper bound on the number of links (i.e. for any agent $i \in N : k_i \leq \kappa$) - this acts as an opportunity cost of linking. The second approach is that an agent i's costs are determined by $c(k_i(\mu))$. The cost function is $c : \mathbb{N}_0 \rightarrow \mathbb{R}$ is positive, strictly increasing and strictly convex function of the degree (i.e. number of links), k_i. Note that unless explicitly stated results are valid under both specifications.

Benefits to agent i is a weighted sum consisting of two elements; network and individual value:

$$b_i(\mu) = \Sigma_{j \neq i} w_{ij}(\mu) \cdot z_{ij} \qquad (1)$$

The network factor, $w_{ij}(\mu)$ is a function of network distance. The individual value factor is z_{ij} which measures the personal value to i of linking to j - the value is a function of the two partners' type $z_{ij} = z(x_i, x_j)$. The function z is assumed twice differentiable as well as takes positive and bounded values.[8] These components are restricted in the subsequent sections.

Transfers and stability This paper explores a static setup where information about agents' types is complete. Any two agents may transfer 'utility' between them. Denote a *net-transfer* in amount of utility from agent j to agent i as $\tau_{ij} \in \mathbb{R}$; thus $\tau_{ij} = -\tau_{ji}$. The matrix of net transfers is denoted τ. For each agent i its payoff including transfers, s_i, is defined as:

$$s_i(\mu, \tau) = u_i(\mu) + \Sigma_{j \in \nu_i(\mu)} \tau_{ij} \qquad (2)$$

$$\Sigma_{i \in N} s_i(\mu, \tau) = U(\mu) \qquad (3)$$

In addition it is assumed that if either of the two linked agents decide to delete their mutual link then the net-transfer between the two agents is set to zero. Note that if two or more agents participate in a coalition move (see definition below) they may respecify mutual transfers.

[8] The upper bound rules out an infinite number of links in equilibrium.

Our notion of stability uses *coalitional moves* from one network μ to another network $\tilde{\mu}$. A move is *feasible* for coalition t if: (i) links added, $\tilde{\mu}\backslash\mu$ are formed only between agents who are both members of coalition t; (ii) links deleted, $\mu\backslash\tilde{\mu}$ have at least one agent in the link who is a member of the coalition t. A coalition $t \in T$ *blocks* a network μ with net-transfers τ if: (i) there is a feasible coalitional move from network μ to network $\tilde{\mu}$ with $\tilde{\tau}$; (ii) for all coalition members the coalitional move leaves them with a higher payoff, i.e., $\forall i \in t : s_i(\tilde{\mu}, \tilde{\tau}) > s_i(\mu, \tau)$.

In this paper we use two equilibrium concepts either *strong stability* where no coalition of any size may block, or the weaker concept, *pairwise (Nash) stability* where at most two agents may coordinate on blocking. The corresponding sets of stable networks are respectively denoted M^{ss} and M^{ps}.[9]

It is worthy noting that strongly stable networks are pairwise stable; thus any condition that applies to pairwise stability also holds for strong stability. In addition without network externalities every pairwise stable network is also strongly stable - see Lemma 1.

Note also that any strongly stable network require efficiency (i.e. maximize aggregate utility). Thus we can employ efficiency to derive structure in strongly stable networks.

3 Sorting in type

This section investigates stratification in type, i.e. based on the agents' exogenous given types. We begin with the setting where network externalities are absent and indirect connections are irrelevant.

In order to derive results a restriction of payoffs is necessary. Let the *(total) link value* be defined as the value of linking for the pair, i.e. $Z_{ij} = z_{ij} + z_{ji}$. Two features of Z are important in our analysis:

Definition 1. The link value has *monotonicity* if $\frac{\partial}{\partial x} Z(x, y) > 0$.

Definition 2. The link value has *supermodularity* if $\frac{\partial^2}{\partial x \partial y} Z(x, y) > 0$; it has *no modularity* when $\frac{\partial^2}{\partial x \partial y} Z(x, y) = 0$. Note that supermodularity entails,

$$Z(x, \tilde{x}) + Z(y, \tilde{y}) > Z(x, y) + Z(\tilde{x}, \tilde{y}), \qquad x > \tilde{y},\, \tilde{x} > y. \tag{4}$$

We investigate a pattern of sorting which is similar to Becker (1973). The pattern of sorting is one where an agent with higher type has partners which weakly dominate in type when compared partner-by-partner with the partners of a lower type agent:

Definition 3. *Top-sorting in type* holds for a subset of networks $\tilde{M}$ if for any network $\mu \in \tilde{M}$ and any two agents i, j where $x_i > x_j$ then $\mathcal{X}(\nu_i(\mu)/\{j\})_l \geq \mathcal{X}(\nu_j(\mu)/\{i\})_l$, $l = 1, .., k_j(\mu)$.

Our first result is that top-sorting in type emerges under similar conditions as in Becker (1973) when network externalities are absent:

Theorem 1. If there is monotonicity, supermodularity and no externalities then top-sorting in type holds for the set of pairwise stable networks M^{ps}.

One important property to note is that in the result above monotonicity of $Z(\cdot)$ is a necessary condition when there are convex costs in addition to supermodularity as in Becker (1973).[10]The

[9] Note our definition of pairwise stability allows substitution of links (simultaneous deletion and formation) - this is stricter than the concept in Jackson and Wolinsky (1996).

[10] From Theorem 3 it is known that this implies that agents degree is weakly monotone in type. Without the weakly monotone relation between type and degree then sorting may fail. This claim can be demonstrated the following stylized

condition of monotonicity can be relaxed when instead there is a degree quota (note the following result requires also a certain distribution of agents):

DEFINITION 4. Let *perfect sorting* hold in a network μ if for all links $ij \in \mu$ it holds that $x_i = x_j$.

REMARK 1. If there is supermodularity, a degree quota κ and a number of agents of each type where $n_x > \kappa$ and $\kappa \cdot n_x$ being even then all pairwise stable networks are perfect sorted.

Note that the pattern prescribed by the above results are not guaranteed when there are externalities. The failure is shown in Example 2 and Corollary 1 in Section 5.

Many partners We proceed to a more general context where indirect connections matter for utility. Whenever we allow for externalities we restrict our attention to network of either:

$$w_{ij}(\mu) = \begin{cases} \delta^{p_{ij}(\mu)-1}, & \text{constant decay} \\ \delta \cdot \mathbb{1}_{\in(1,\infty)}(p_{ij}(\mu)), & \text{hyperbolic decay} \end{cases} \tag{5}$$

where $\mathbb{1}_{\in(1,\infty)}(l)$ is the Dirac measure/indicator function of whether $1 < l < \infty$. Constant decay corresponds to benefits in the 'connections-model' from Jackson and Wolinsky (1996). Note that hyperbolic decay is a special case where there is no decay beyond that at distance of two.

First we investigate what pattern of linking is exhibited in the case of (asymptotic) infinite number of potential partners, i.e. a "large" market. We investigate perfect sorting in large markets which is when the share of links between agents of same type converges to one:

DEFINITION 5. Let *asymptotic perfect sorting* hold for a sequence of networks sets $\tilde{M}_n$ if for any network $\mu \in \tilde{M}_n$ where $n \to \infty$ it holds that $|\{ij \in \mu : \, x_i = x_j\}|/|\mu| \simeq 1$.

Our result on sorting in large markets is demonstrating sufficient conditions for asymptotic perfect sorting to emerge in strongly stable networks:

PROPOSITION 1. If there is supermodularity, a degree quota and constant decay with asymptotic independence ($\delta < (\kappa - 1)^{-1}$) then there is asymptotic perfect sorting for strongly stable networks.

The result above demonstrates that the availability of many agents for linking induces perfect sorting in strongly stable networks.[11] This is the same prediction as the conclusion of Becker (1973)'s marriage market but holds in the presence of externalities with constant decay. Note that the result relies on efficiency requiring perfect sorting and using this to infer properties of strong stability.

In the remainder of this section we treat the more common case of a limited number of potential partners. When numbers of of links are limited we show that the previously established patterns of sorting may be inefficient.

Over-sorting, simple example In this subsection and the following we reexamine our results on sorting by illustrating a class of sorted networks that are suboptimal when introducing network externalities. The sub-optimality holds despite fulfilling supermodularity (which is necessary and sufficient condition for sorting to be stable and optimal under certain conditions cf. Remark 1).

example. Suppose there are two agents with high type $\bar{x}$ and one of low type $\underline{x}$. Suppose we have that the high types have a degree quota of one, but the low has two. Then the high types will link only if $Z(\bar{x}, \bar{x}) > Z(\bar{x}, \underline{x}) + Z(\bar{x}, \underline{x})$.

[11] In network science a measure of assortative behavior in networks is the coefficient of assortative mixing. Perfect sorting corresponds to a coefficient =1, cf. Newman (2003).

Excessive sorting refers to a network with perfect sorting according to type where the segregated groups could collectively benefit from connecting. However, they fail to connect as incentives do not internalize externalities under pairwise network formation. The formal definition is:

Definition 6. Denote the set of *sub-connected* networks of μ as $M^{sub-conn}(\mu)$ and be defined as:

$$M^{sub-conn}(\mu) = \{\tilde{\mu} \in M : \exists i, i', j, j' \in N \text{ s.t. } \tilde{\mu} = \mu \cup \{ij, i'j'\} \backslash \{ii', jj'\} \text{ and } x_i \neq x_j\}$$

Definition 7. A network μ is *over-sorted* if it has perfect sorting and there exist thresholds $\underline{\delta} < \bar{\delta}$ such that for all $\delta \in (\underline{\delta}, \bar{\delta}]$ it holds that (i) μ is pairwise (Nash) stable for some transfers τ, and; (ii) any network $\tilde{\mu} \in M^{sub-conn}(\mu)$ provides higher aggregate utility than μ.

In order to demonstrate the idea the example below introduces over-sorting in a stylized, simple example. The example is represented in Figure 1.

Example 1. There are six agents; three high type (agent 1,2,3) and three low type (agent 4,5,6). Moreover, there is supermodularity, degree quota (κ=2) and constant decay. Define two networks:

- a network with sorting in type, $\mu = \{12, 13, 23, 45, 46, 56\}$, see Figure 1.A;
- a network which is sub-connected, $\tilde{\mu} = \{12, 23, 34, 45, 56, 61\}$, see Figure 1.C.

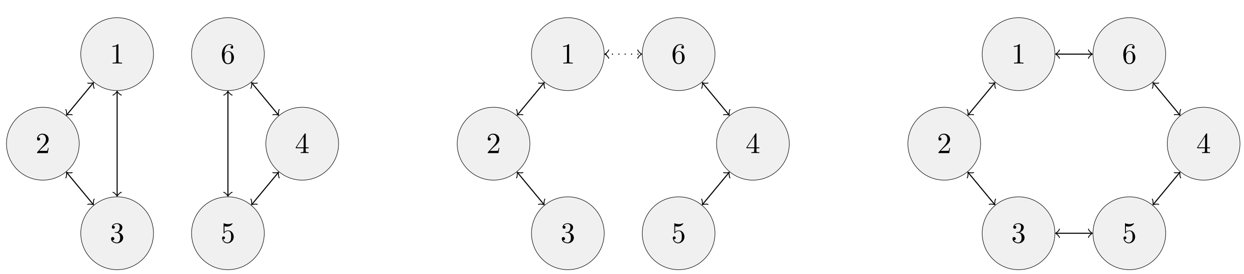
We show in the following that network μ has over-sorting. That is, we show for a range of decay-factors that μ is inefficient (compared with $\tilde{\mu}$), and μ is pairwise stable for absence of transfers.

Under pairwise stability at most one link can be formed in a single move. Without transfers all formed links have value and thus deletion of a link always leads to a loss. Thus only coalitional moves where new links are formed can be valuable. For network μ all links to same types are already formed. Therefore the only feasible and relevant move involves forming a link across types e.g agents $1, 6$ deleting a link with 3 and 4 respectively to form a link together; this network is denoted $\hat{\mu} = \mu \cup \{16\} \backslash \{13, 46\}$, see Figure 1.B. Benefits for agents 1 and 6 from network μ or deviating are:

$$\begin{aligned} u_1(\hat{\mu}) + u_6(\hat{\mu}) &= (1+\delta) \cdot [z(\bar{x}, \bar{x}) + z(\underline{x}, \underline{x})] + [1 + \delta + \delta^2] \cdot [z(\bar{x}, \underline{x}) + z(\underline{x}, \bar{x})], \\ &= (1+\delta) \cdot \tfrac{1}{2} \cdot [Z(\bar{x}, \bar{x}) + Z(\underline{x}, \underline{x})] + [1 + \delta + \delta^2] \cdot Z(\bar{x}, \underline{x}), \\ u_1(\mu) + u_6(\mu) &= 2 \cdot [z(\bar{x}, \bar{x}) + z(\underline{x}, \underline{x})] = Z(\bar{x}, \bar{x}) + Z(\underline{x}, \underline{x}). \end{aligned}$$

The condition for deviation to $\hat{\mu}$ not to be pairwise profitable is $u_1(\mu) + u_6(\mu) > u_1(\hat{\mu}) + u_6(\hat{\mu})$; note this condition is sufficient for pairwise stability due to payoff symmetry in μ for no transfers.

We now turn to deriving condition for when segregating is inefficient. The aggregate benefits over



(A): Segregated network, μ. Pairwise stable if $\delta \leq \bar{\delta}(\hat{Z})$.

(B): Pairwise move by agent 1, 6 who form link and remove one each.

(C): Connected network, $\tilde{\mu}$. Efficient if $\delta \geq \underline{\delta}(\hat{Z})$.

Figure 1: The above three networks depict Example 1.

all agents of the two networks μ and $\tilde{\mu}$ is expressed below in the two equations.

$$\begin{aligned} U(\tilde{\mu}) &= (2+\delta)\cdot[Z(\bar{x},\bar{x})+Z(\underline{x},\underline{x})]+\left[2+4\delta+3\delta^2\right]\cdot Z(\bar{x},\underline{x}) \\ U(\mu) &= 3\cdot[Z(\bar{x},\bar{x})+Z(\underline{x},\underline{x})] \end{aligned}$$

Sorting is inefficient when: $U(\mu) < U(\tilde{\mu})$. The two quadratic inequalities governing pairwise stability and inefficiency has the following positive solution:

$$\begin{aligned} \underline{\delta} &= -\tfrac{1}{3}\left(\hat{Z}+2\right)+\tfrac{1}{3}\sqrt{(\hat{Z}+2)^2+6(\hat{Z}-1)}, \\ \bar{\delta} &= -\tfrac{1}{2}\left(\hat{Z}+1\right)+\tfrac{1}{2}\sqrt{(\hat{Z}+1)^2+4(\hat{Z}-1)}. \end{aligned} \qquad \hat{Z}=\frac{Z(\bar{x},\bar{x})+Z(\underline{x},\underline{x})}{2Z(\bar{x},\underline{x})}$$

The example above demonstrates that sorting can be inefficient when there are network effects despite presence of supermodularity. The inefficiency stems from a novel source - the pairwise formation of links. The intuition is that under pairwise deviation the two agents do not internalize the total value created for other agents number of indirect links between a high and a low agent.

Over-sorting, generalization In the following subsection we elaborate on the example from the previous subsection and generalize it for first hyperbolic decay then constant decay. As we proceed with our generalization we introduce new concepts to convey the results.

A *component*, $\tilde{\mu}$ of network μ is a subnetwork (i.e. $\tilde{\mu}\subseteq\mu$) where the following holds. For any agent $i\in N(\tilde{\mu})$, where $N(\tilde{\mu})=\{i\in N:\exists j \text{ s.t. } ij\in\tilde{\mu}\}$, it holds that: (i) agent i is connected in $\tilde{\mu}$ to all other agents $j\in N(\tilde{\mu})$; (ii) for all $j\in N$ then $ij\in\mu$ implies $ij\in\tilde{\mu}$. Let the subset of potential networks which are perfectly sorted into one component for each type be denoted $M^{sort:comp}$.

A *cycle* is a network where all agents has two links and are connected. E.g. for $N=\{i_1,i_2,...,i_n\}$ a cycle is $\{i_1i_2,i_2i_3,...,i_ni_1\}$. A *cyclic* network μ has a subnetwork $\tilde{\mu}\subseteq\mu$ where $\tilde{\mu}$ is a cycle. Note a network where degree quota is even and binding for all agents is cyclic cf. Veblen (1912).[12]

Our next result shows that over-sorting holds under hyperbolic decay under the additional restrictions that components are cyclic and degree quota binds for all agents:

Proposition 2. Suppose there is supermodularity, hyperbolic decay and a degree quota, it follows that for any network μ in $M^{sort:comp}$ if μ satisfies both a binding degree quota (i.e. $\forall i: k_i(\mu)=\kappa$) and having each component $\tilde{\mu}\subseteq\mu$ being cyclic then μ has over-sorting as follows:

$$\underline{\delta}\le\min_{x,\tilde{x}\in X}\left(\frac{\hat{Z}_{x,\tilde{x}}-1}{\hat{Z}_{x,\tilde{x}}-1+\frac{1}{2}n_x\cdot n_{\tilde{x}}}\right) \text{ and } \bar{\delta}\ge\min_{x,\tilde{x}\in X}\left(\frac{\hat{Z}_{x,\tilde{x}}-1}{\max(n_x,n_{\tilde{x}})+\hat{Z}_{x,\tilde{x}}-1}\right), \quad \hat{Z}_{x,\tilde{x}}=\frac{Z(x,x)+Z(\tilde{x},\tilde{x})}{2Z(x,\tilde{x})}$$

We proceed with generalizing over-sorting under constant decay - this is done for a subclass of perfect sorted networks. For this class of networks it is possible to demonstrate that over-sorting persists as a general phenomenon. Informally put, the relevant class of networks has the property that from the perspective of every agent each network appears as a tree when disregarding the agents furthest away.[13] Thus these networks are called local trees. The definition depends on the *diameter* of a network which for network μ is $m(\mu)=\sup_{i,j\in N}p_{ij}(\mu)$.

Definition 8. A network μ is a *local tree* when each agent i has κ links such that for each other agent $j\neq i$ at distance $p_{ij}(\mu)\le m(n,\kappa)-2$ there are $\kappa-1$ links between agent j and j' such that $p_{ij}(\mu)=p_{ij'}(\mu)-1$, and the diameter is finite and defined as follows

$$m_{n,\kappa}(\mu) = \arg\min_m\{m:\Sigma_{l=1}^m(\kappa(\kappa-1)^{l-1})+1\ge n\} \tag{6}$$

[12]Veblen (1912) shows that the network can be decomposed into a union of disjoint cycles.

[13]A tree is network where every pair of agents are connected by a unique path.

The structure of local trees entails that each agent has $\kappa \cdot (\kappa-1)^{p-1}$ agents at distance $p < m$, where $m = m_{n,\kappa}(\cdot)$. At distance $p = m$ there are $n - \sum_{l=1}^{m-1} \kappa \cdot (\kappa-1)^{l-1}$ (all remaining agents). This structure implies that every agent's utility is maximized subject to the constraint of all agents having at most κ links;[14] thus utility before transfers is symmetric.

When a local tree network fulfills $n = \sum_{l=1}^{m} \kappa \cdot (\kappa-1)^{l-1}$ then it is an *exact local tree*. There are two subclasses of exact local trees which are worth mentioning. The first is a cycle network which have a minimal degree quota ($\kappa = 2$) among local trees and a maximal diameter ($m = \lceil \frac{n-1}{2} \rceil$). The second is a *clique* where all agents are linked, i.e. the complete network. Cliques have maximal degree quotas ($\kappa = n-1$) and minimal diameters ($m = 1$). Both subclasses has a network which exists for any n. Note that in Example 1 each of the two components is both a cycle and a clique.

A necessary condition for local trees to exist is that the degree binding (i.e. $\forall i \in N : k_i = \kappa$) - this condition is only possible when $n \cdot \kappa$ is even.

In order to derive our results it is necessary to restrict the set of local trees:[15]

Definition 9. A local tree μ has *symmetric losses* when at every distance $p \in \mathbb{N}$ it holds that $|\{i \in N : p_{\iota i}(\mu \backslash \{\iota\iota'\}) = p\}| = |\{i \in N : p_{\iota' i}(\mu \backslash \{\iota\iota'\}) = p\}|$.

Our generalization of over-sorting under constant decay is expressed below. The theorem is applicable to numerous settings where social networks are formed such as schools or in the labour market. These institutions may pre-sort individuals according to talent or otherwise which may lead to a sorted network which is stable but inefficient. A specific example could be a school where sorting by academic performance type is common in many countries and is often known as tracking. In such cases the sorting induced by the institution could lead to a stable network with no linking across despite links across having potential to raise welfare.

Theorem 2. When there is supermodularity, a degree quota κ and each type has identical number of agents then networks segregated into local trees with symmetric losses are over-sorted.

For constant decay over-sorting thresholds $\underline{\delta}, \bar{\delta}$ can be determined explicitly by solving polynomial equations for every deviation. Moreover, for exact local trees there is a unique solution. In Figure 2 the two thresholds from Theorem 2, $\underline{\delta}(\hat{Z}), \bar{\delta}(\hat{Z})$. The plots are made for variations of exact local trees. The upper plots corresponds to cliques with various sizes. The lower plot have fixed degree quota (κ=100) and the threshold is simulated using pattern in utility that is demonstrated in Appendix C. The plots show the scope for inefficiency, i.e. the gap between $\underline{\delta}(\hat{Z}), \bar{\delta}(\hat{Z})$, increases with the number of agents involved. This makes sense intuitively as the two agents forming the link will fail to account for an increasing number of indirect connections between the two groups. As the number of indirect connections increases at with the squared with total number of agents then larger populations will lead to larger gaps of inefficiency.

The remainder of this subsection will sketch a policy intervention that may help overcome over-sorting. The networks with over-sorting are possible to reshape by providing some agents with incentives to link across types.

Definition 10. A *network-contingent contract* for a given agent is a mapping from every potential network M to a transfer for the agent.

[14] The maximization of utility follows from the observation that each agent has at most κ links, so at distance p there can be at most $\kappa \cdot (\kappa-1)^{p-1}$ agents.

[15] We are uncertain whether symmetric losses is a generic property for all local trees as we have failed to prove it. However, in simulations it has been shown to be valid in the set of all local trees up to size $n = 10$ and for size up $n = 16$ it has been shown for every network we have examined.

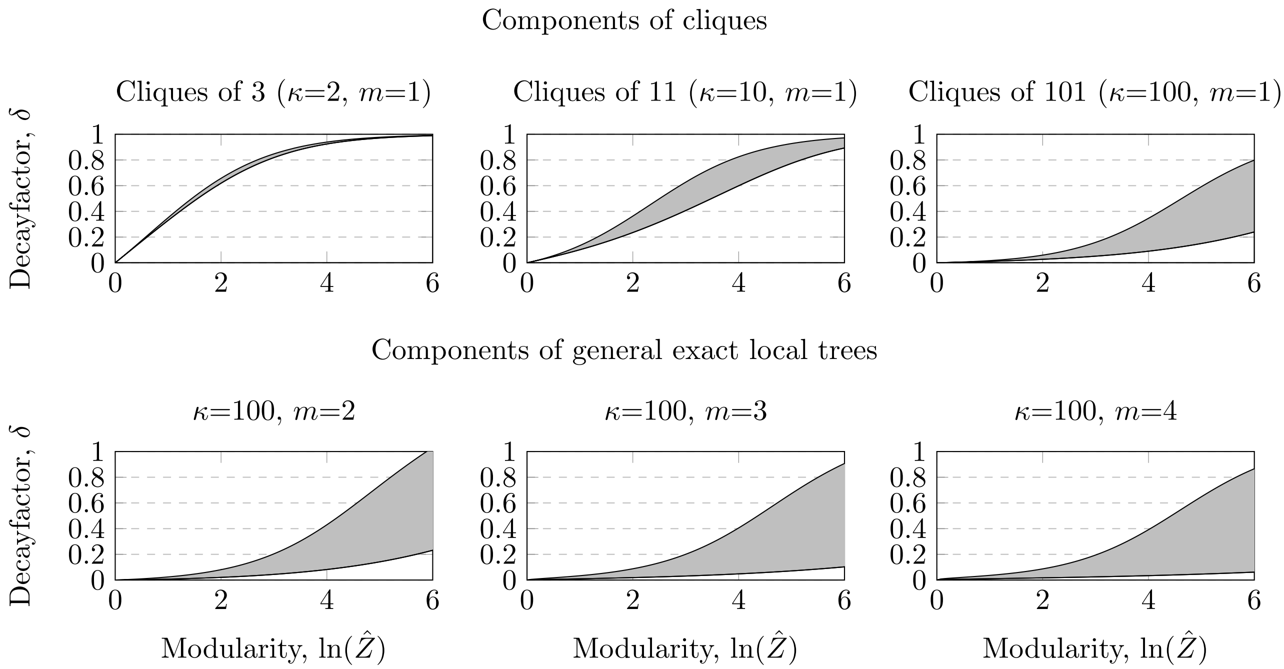


Figure 2: Visualization of thresholds for connecting from Theorem 2 for components of cliques or exact local trees (where thresholds stem from Equations 18, 19, 22, 23).

Proposition 3. If network-contingent contracts exist then a policy maker can incentivize agents to deviate pairwise and form a network with higher welfare from an over-sorted network.

Proof. Let μ have over-sorting and let $\delta \in (\underline{\delta}, \overline{\delta}]$. By construction there exists a network $\tilde{\mu}$ which has higher aggregate utility and is sub-connected network of μ; let the two pairs of agents ii', jj' be agents such that $\tilde{\mu} = \mu \cup \{ij, i'j'\} \backslash \{ii', jj'\}$.

Let $\hat{\mu} = \mu \cup \{ij\} \backslash \{ii', jj'\}$. As μ is pairwise stable (by definition) it follows that: $b_i(\mu) - b_i(\hat{\mu}) + b_j(\mu) - b_j(\hat{\mu}) \leq 0$. Specify a network-contingent contract to i, j as follows:

- if $\hat{\mu}$ is formed they receive a transfer of $\frac{1}{2}\varepsilon$ each;
- if $\tilde{\mu}$ is formed i receives $\max(0, \frac{1}{2}\varepsilon + b_i(\hat{\mu}) - b_i(\tilde{\mu}))$ and j gets $\max(0, \frac{1}{2}\varepsilon + b_j(\hat{\mu}) - b_j(\tilde{\mu}))$;
- else the transfer is zero.

When it holds that $\varepsilon \in (b_i(\mu) - b_i(\hat{\mu}) + b_j(\mu) - b_j(\hat{\mu}), U(\tilde{\mu}) - U(\mu))$ then i, j are a blocking coalition that can gain by deviating to $\hat{\mu}$.

In network $\hat{\mu}$ agents i', j' will have an incentive to form a link with one another as they both have surplus link capacity (each has a degree which is one below the degree quota) and forming a link always provide value as it shortens paths lengths. Thus there are two pairwise moves from μ to $\hat{\mu}$ and from $\hat{\mu}$ to $\tilde{\mu}$ which both provide strictly higher utility to the deviating agents. □

Note that the individual compensation for connecting paid to agents may not be equal. In particular one could in the case where there is both supermodularity and monotonicity in $Z(\cdot)$ then agents may receive compensation proportional to their type; this would be the case in an over-sorted network with no transfers where components have same number of agents and are isomorphic to another.

4 Network structure

In this section we examine the structure of stable networks when measured by specific measures of network centrality. The measures of centrality are degree and some abstract measure of centrality.

The first question posed is the following: what is the relation between network measures of centrality and type? The short answer is that type and connectivity go hand in hand under certain circumstances - higher type implies higher centrality. We call this degree monotonicity which we use in our result:

Definition 11. A set of networks $\tilde{M}$ has *degree monotonicity* in type if:

$$\forall \mu \in \tilde{M}: \ x_i > x_j \Rightarrow k_i(\mu) \geq k_j(\mu).$$

Theorem 3. Suppose there are no externalities and monotonicity in link value then the set of pairwise stable networks has degree monotonicity.

The intuition behind the above result is that the marginal benefits for forming an additional link is strictly increasing in talent. Thus a higher talented agent has an increased incentive to form links.

In the following paragraphs we investigate the situation where network externalities affect indirect relations. It turns out that the pattern of monotonic centrality in talent from Theorem 3 may be strengthened to hold for more general centrality measures that take into account indirect friends. But the strengthening requires restrictions on externalities and the equilibrium concept. We define monotonicity of δ-decay centrality by substituting the measure in the definition of degree monotonicity. Our result is:

Proposition 4. Suppose there are externalities as well as monotonicity and no modularity then the set of strongly stable networks has δ-decay monotonicity.

The result above emerges from the fact that there are only monotone and independent effects of more talent (due to the absence of modularity). The independence of effects entails that the link value can be split into a part for each partner, $Z(x, \tilde{x}) = \hat{Z}(x) + \hat{Z}(\tilde{x})$. It is possible to show that a given agent i's total contributions to the network is the product between $\hat{Z}(x_i)$ and the sum of network weights. Under these circumstances monotonicity implies that more talented agents must have higher measure of network weights or else efficiency implied by strong stability is violated.

We conclude our analysis of monotonic centrality by pointing to the limitations of our results. Firstly when the requirement of no modularity is violated for instance by having supermodularity then Proposition 4 may not hold. This is seen in example 3 which is found in Appendix E. Secondly by relaxing the requirement of strong stability in Proposition 4 to pairwise stability it follows that monotonic centrality also fails. This is seen from Example 2 and Proposition 5.

Sorting in degree The question we investigate in the remainder of this section is whether or not there is sorting in degree, i.e. does more gregarious agents (agents with many partners), tend to associate with agents who are as gregarious as themselves? The simple answer is yes, under certain circumstances elaborated below.

We begin with an analysis of the setup where externalities are absent. This part of the analysis builds on the two previous sections' results: firstly, there is a positive relationship between higher talent and (weakly) more partners (Theorem 3), and; secondly, partners are chosen assortative in talent (Theorem 1). By combining these two previously established results it can be shown there will also be degree assortativity. Although the intuition of combining the two previous results is straightforward additional structure needs to be imposed to show the result. Let *complete heterogeneity* be satisfied if for every agent has a unique level of talent, i.e. $n = |X|$.

The concept of degree sorting we're using is analogous to top-sorting defined for sorting patterns; i.e. every agent with strictly higher degree must have partners which weakly dominate in degree when compared partner-by-partner with the partners of agent with lower degree:

Definition 12. A set of networks $\tilde{M}$ has *top sorting in degree* when:

$$\forall \mu \in \tilde{M},\ \forall i,j \in N, k_i(\mu) > k_j(\mu): \qquad \mathcal{K}(\nu_i(\mu)/\{j\})_l \geq \mathcal{K}(\nu_j(\mu)/\{i\})_l, \qquad l = 1, ..., k_j(\mu).$$

Theorem 4. If there are no externalities, supermodularity and monotonicity in link value, as well as complete heterogeneity then the set of pairwise stable networks has top sorting in degree.

The above result is identical to Theorem 1 with the exception that the assortative property in this result is degree and this result has an added requirement. The emergence of this assortative pattern is not only a theoretical curiosity - this pattern is an underlying property observed across various empirical social networks, see the literature review.

The feature emerging degree assortativity in Theorem 4 is relevant as it indicates that positive degree assortativity may indicate people sort themselves according to talent.

When we move to a setting of externalities our pattern of sorting by degree does not hold in general. The reason is that either sorting by talent or monotonic degree may fail. This is captured by the follows from Proposition 5 in the next section as it shows both sorting in type and monotonic centrality may fail.

5 Limits to sorting and centrality monotonicity

In the previous two sections we derived patterns for sorting and network centrality which we address again in this section. We show the failure of those patterns is possible despite despite fulfilling the conditions that are sufficient in the previous sections: monotone and supermodular link value. The failure emerges for a generic network where the agent with least value is most central has pairwise stability - this constitutes a lack of sorting and non-monotonic centrality in type.

The generic network we work with in this sections also lacks efficiency. The inefficiency stems from the fact that agents can be ranked by in terms of their contributions to a link under monotonicity of link value. This implies the agent with least value is more central than other agents and this by subtituting this agent with others then aggregate value could be improved. The reason is that high value agents fail to coordinate on displacing the low value agent with a high value agent which would make them all better off. This situation is illustrated in the following example with three agents:

Example 2. Let there be three agents such that two agents (i=1,2) have high types, $\bar{x}$, and the last agent (i=3) has a low type, $\underline{x}$. Suppose additionally there is at both weak supermodularity but (strict) monotonicity holds. Moreover network externalities is based on shortest paths as in Equation 5 with constant or hyperbolic decay. Finally let there be convex costs.

Let network μ have a structure such that agent 3 is the center of a star-network, i.e. $\mu = \{13, 23\}$. The network μ is inefficient as $\mu = \{12, 13\}$ provides strictly higher aggregate utility as for $\delta \in [0, 1)$:

$$\begin{aligned} U(\tilde{\mu}) &> U(\mu) \\ Z(\bar{x}, \bar{x}) + (1+\delta) \cdot Z(\bar{x}, \underline{x}) &> \delta \cdot Z(\bar{x}, \bar{x}) + 2Z(\bar{x}, \underline{x}) \\ (1-\delta) \cdot Z(\bar{x}, \bar{x}) &> (1-\delta) \cdot Z(\bar{x}, \underline{x}) \end{aligned}$$

We show below that network μ may be pairwise stable for some transfers. We begin with conditions for when deviating is unprofitable when only are deleted links (no links formed). Cutting all links corresponds to an outside option with net-utility of zero - this is Inequality 7. Another option is agent 3 deleting one link and getting net value of the remaining link - see Inequality 8:

$$s_i(\mu,\tau) \geq 0, \qquad i \in N, \tag{7}$$
$$\tau_{3i} \geq \Delta c(1) - z(\underline{x},\bar{x}), \qquad i \in \{2,3\} \tag{8}$$

It remains to be shown the condition that prevent any pairwise deviation by forming a link. Only agent 1,2 can form a link. There are three options for forming link 12: deleting no links, deleting one link or deleting all - these are captured by respectively Inequality 9, 10 and 11:

$$2 \cdot \Delta c(2) \geq (1-\delta) \cdot Z(\bar{x},\bar{x}) \tag{9}$$
$$s_1(\mu,\tau) + s_2(\mu,\tau) \geq Z(\bar{x},\bar{x}) + z(\bar{x},\underline{x}) \cdot (1+\delta) - c(1) - c(2) + \max\{\tau_{13},\tau_{23}\} \tag{10}$$
$$s_i(\mu,\tau) + s_j(\mu,\tau) \geq Z_{ij} - 2 \cdot c(1), \qquad i \neq j \tag{11}$$

We conclude the example by mentioning that the above conditions are possible to satisfy. Lemma 2 provides sufficient conditions for Inequality (7)-(11) to be satisfied; the conditions are (near) linear costs ($\tilde{c}$ of forming a link) and δ being sufficiently high along with the two inequalities:

$$2 \cdot Z(\bar{x},\underline{x}) + Z(\bar{x},\bar{x}) \geq 4\tilde{c} \quad \text{and} \quad Z(\bar{x},\underline{x}) > \tilde{c}.$$

Example 2 points to a more general result - sorting in type may not hold - even for many agents. This failure is expressed in the following result:[16]

Proposition 5. For any number of agents if there are externalities, monotonicity, supermodularity and a cost function then the network where the least type is center of a star is inefficient but pairwise stable under certain conditions.

As alluded to previously the proposition has a series of implications for patterns in the network which are captured in the following corollary:

Corollary 1. When there are externalities then under pairwise stability:
- sorting in type as outlined by Theorem 1 may fail;
- centrality monotonicity in type as described by Theorem 3 and Proposition 4 may fail.

[16] The proof relies on the auxiliary results Appendix G: Lemma 3 and 4.

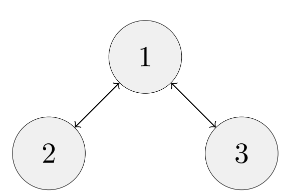


Figure 3: The above network depicts the limits of sorting and monotonic centrality from Example 2. The low value agent (i=3) is excessively central which is inefficient. Moreover there is insufficient sorting as the high value agents (i=1,2) are not linked.

The corollary above demonstrates that despite meeting the conditions of Becker (1973) these conditions are not sufficient for sorting in social networks with externalities. The lack of sorting is a market failure of where the worst candidate among the group has been chosen to be the center. Note that this result can be seen as analogue to Farrell and Saloner (1985) and Katz and Shapiro (1985) in markets of buyers and sellers as well as (network) externalities.

6 Conclusive discussion

Our analysis is based on strict assumptions which are discussed in this final section. We avoid discussing sorting under search as there is a large literature e.g. Shimer and Smith (2000).

The most severe caveat with our analysis, and stable networks in general, is that these networks may not exist cf. Gale and Shapley (1962). Note that the scope in lack of existence may not be excessive. We can derive the cost of stability (the necessary payments to induce stability) from Bachrach, Elkind, Meir, Pasechnik, Zuckerman, Rothe, and Rosenschein (2009).

There is also a number of restrictive assumptions on payoff. The most crucial assumption is that payoffs are separable for each link. The separability closes the model to a variety of possible peer effect specifications, e.g. utility from groups. Another stylized assumptions is the adaption of Jackson and Wolinsky (1996)'s "connection model" which required externalities from indirect connections to be discounted with geometric decay in network distance. Nevertheless by removing separability and allowing for non-geometric decay many of the results are likely to be robust although more difficult to establish. Two critical assumptions are supermodularity and monotonicity along with perfect transferability. Nevertheless, as mentioned in the introduction, these two assumptions can be replaced by monotonicity in individual link value and perfect non-transferability.[17] As noted in the description of methodology, the monotonicity of individual link value is consistent with research on peer effects. Finally the assumptions on cost of linking implied that effects of effort, such as in the coauthor model of Jackson and Wolinsky (1996) are absent. However, Baccara and Yariv (2013) shows that for some classes of effort then assortative matching into groups can occur.

Finally it is important to note that the derivation of Proposition 4 and Proposition 1 rely on efficiency is necessary for strong stability. Although efficiency is a unique property for strong stability (and does not hold for weaker concepts) then strong stability should be seen as a refinement with desirable properties which makes it more likely when it exists. In some circumstances the existence of contracts where an agent may subsidize or penalize another agent's link formation with alternative agents may imply that strong stability even if contracts were limited to being pairwise specified, cf. Bloch and Jackson (2007).

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[17] Or the generalized increasing in differences condition which allows for a setting of mix between transferable and non-transferable utility, cf. Legros and Newman (2007).

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A Auxiliary results

LEMMA 1. In the absence of no network externalities then the set of strongly stable networks is equivalent to the set of pairwise stable networks, i.e., $M^{ps}_{\delta=0} = M^{ss}_{\delta=0}$.

Proof. By definition it holds that $M^{ps}_{\delta=0} \subseteq M^{ps}_{\delta=0}$, thus we need to show that $M^{ps}_{\delta=0} \subseteq M^{ps}_{\delta=0}$ to prove the claim. This claim is shown using similar to arguments to Klaus and Walzl (2009)'s Theorem 3.i.

Let μ with associated contracts τ be a network which is blocked by a coalition. It will be shown that for every coalition $t \in T$ that blocks, within the coalition there is a subset of no more than two members that also wishes to block the network. Let $\tilde{\mu}$ be the alternative network that the blocking coalitions implements through a feasible coalition move and τ be the transfers associated with $\tilde{\mu}$.

It is always possible to partition the set of deleted links $\mu\backslash\tilde{\mu}$ into two: (i) a subset denoted $\hat{\mu}$ where for each link ij that can be deleted where one of the two partners can benefit, i.e. it holds that either $z_{ij} + \tau_{ij} - [c_i(\mu) - c_i(\mu\backslash ij)] < 0$ or $z_{ji} + \tau_{ji} - [c_j(\mu) - c_i(\mu\backslash ij)] < 0$; (ii) a subset denoted $\breve{\mu}$ where for each link ij neither of the previous two inequalities are satisfied.

Suppose that the first partition is non-empty, i.e. $\hat{\mu} \neq \emptyset$. However, as deleting links can be done by a single agent on its own then the move only takes needs the coalition of that agent to delete the link. Thus any part of a coalitional move that only involves profitably removing links can be performed in parts by a coalition with a single agent - therefore this move is also a pairwise block.

Thus it remains to be shown that the remaining part of coalitional move also can be performed as a pairwise block, i.e. when forming $\tilde{\mu}\backslash\mu$ and deleting $\breve{\mu}$. This part of the coalitional move must entail forming links as no links can be deleted profitably. The set of formed links $\tilde{\mu}\backslash\mu$ can be partitioned into a number of $|\tilde{\mu}\backslash\mu|$ feasible submoves of adding a single link while deleting links by each of the agents i and j who form a link. The feasibility for each of the partitioned moves is always true when there is a cost function as moves are unrestricted. It is now argued that each of the partitioned moves are feasible when there is a degree quota. If the network $\mu \cup ij$ is feasible then the move of simply adding the link is feasible. If $\mu \cup ij$ is not feasible, then agents i and j can delete at most one link each and if both μ and $\tilde{\mu}$ are feasible then this also feasible as the degree quota is kept.

For the coalitional move to $\tilde{\mu}$ it must be that at least at least one link among the implemented links $\tilde{\mu}\backslash\mu$ has a strictly positive value that exceeds the loss from deleting at most one link for each of two agents forming the link. This follows as it is known that deleting one or more links cannot add any value and thus must have weakly negative value and that by definition the total value to the blocking coalition must be positive. As every one of the partitioned moves is feasible, it follows that for every coalitional move there are two agents who can form link while potentially destroying current links and both be better off. In other words, for every coalition that blocks, there is a pairwise coalition that blocks. □

FACT 1. For every κ, n such that $n > \kappa$ and $n \cdot \kappa$ is even there exists a network $\mu_{n,\kappa}$ where all agents have exactly κ neighbors.

Proof. Suppose n is even. Let % be the modulus operator. We can construct the following networks.

$$\begin{aligned} \hat{\mu}_{n,\kappa} &= \left\{ij : i \in \{1, ..., \tfrac{n}{2}\}, j \in \{(\tfrac{n}{2} + i \,\%\, \tfrac{n}{2}), ..., (\tfrac{n}{2} + [i + \kappa - 1] \,\%\, \tfrac{n}{2}\})\right\}, \ \kappa \leq \tfrac{n}{2}, \\ \tilde{\mu}_{n,k} &= \begin{cases} \hat{\mu}_{n,\kappa}, & \kappa \leq \frac{n}{2} \\ \mu_{n,\kappa} = \mu_c \backslash \hat{\mu}_{n,n-\kappa-1}, & \kappa > \frac{n}{2} \end{cases} \end{aligned}$$

Letting $\mu_{n,\kappa} = \tilde{\mu}_{n,\kappa}$ is sufficient for n is even. When n is odd we know that κ is even and thus we can use the following amended procedure instead:

$$\begin{aligned} \mu_{n,\kappa} &= \tilde{\mu}_{n-1,\kappa}\backslash\left\{ij:\, i\le\tfrac{\kappa}{2},\, j=i+\tfrac{\kappa}{2}\right\}\cup\left\{ij:\, i=n,\, j\in\{1,..,\tfrac{\kappa}{2}\}\cup\{\tfrac{n+1}{2},..,\tfrac{n-1+\kappa}{2}\}\right\},\ \kappa\le\tfrac{n-1}{2} \\ \mu_{n,\kappa} &= \tilde{\mu}_{n-1,\kappa}\backslash\{ij:\, i\in\{1,3,...,\kappa-1\},\, j=i+1\}\cup\{ij:\, i=n,\, j\le\kappa\},\ \kappa>\tfrac{n-1}{2}. \end{aligned}$$

□

B Sorting in talent

Theorem 1: If there is monotonicity, supermodularity and no externalities then top-sorting in type holds for the set of pairwise stable networks M^{ps}.

Proof. Suppose the claim is false. Let q be the lowest index for which the condition fail: for all $l < q$ it holds that $\mathcal{X}(\nu_i(\mu)/\{j\})_l \geq \mathcal{X}(\nu_j(\mu)/\{i\})_l$. Thus there are two agents i', j' such that:

$$\begin{aligned} x_{j'} &= \mathcal{X}(\nu_j(\mu))_q, \quad j' \in (\nu_j(\mu)\backslash(\nu_i(\mu)\cup\{i\})), \\ x_{i'} &< \mathcal{X}(\nu_j(\mu))_q, \quad i' \in (\nu_i(\mu)\backslash(\nu_j(\mu)\cup\{j\})). \end{aligned}$$

The argument why $i' \notin (\nu_j(\mu)\cup\{j\})$ is that the number of agents in $\nu_i(\mu)$ with above index q is higher for agent i than for agent j (follows from Theorem 3); thus such an agent must exist

Thus it holds that $x_i > x_j, x_{i'} < x_{j'}$ along with $ij', ji' \notin \mu$. However, this fact implies that there is a violation of strong stability. This follows as agents i, i', j, j' can deviate by destroying $\{ij, i'j'\}$ and forming $\{ij', i'j\}$ and thus increase payoffs due to supermodularity (cf. Equation 4). From Lemma 1 it follows that pairwise stability is also violated if strong stability is violated. □

Remark 1: If there is supermodularity, a degree quota κ and a number of agents of each type where $n_x > \kappa$ and $\kappa \cdot n_x$ being even then all pairwise stable networks are perfect sorted.

Proof. Let $N_{x,\hat{x}}(\mu) = |\{ij \in \mu :\, x_i = x, x_j = \hat{x}\}|$. From any network $\mu \in \mathcal{M}$ we can construct $\hat{\mu}(\mu)$ such that $N_{x,\hat{x}}(\hat{\mu}) \geq N_{x,\hat{x}}(\mu)$. $\hat{\mu}$ is created as follows:

1. For every x create a subnetwork of only agents with type x where degree quota is binding from Fact 1 in Appendix A. Denote the subnetwork for type x as μ_x; this subnetwork has $\frac{\kappa\cdot n_x}{2}$ links.
2. For every x delete $\hat{n}_x = \frac{n_x\cdot\kappa - \sum_{\hat{x}\neq x} N_{x,\hat{x}}(\mu)}{2}$ links from μ_x such that the difference in degree is at most one between agents of type x (this can be using pattern from Fact 1). let the subnetwork created by this procedure be denoted $\tilde{\mu}_x$.
3. For every x and $\hat{x} < x$: form $N_{x,\hat{x}}(\mu)$ links between agents of type x and type $\hat{x}$ - this must feasible as there are no links between the two types in $\cup_{x\in X}\tilde{\mu}_x$ and the number is feasible for μ - denote the set of newly assigned links; assign the links such that each step $x, \hat{x}$ such the difference in degree in $(\cup_{x\in X}\tilde{\mu}_x)\cup(\cup_{\tilde{x}>x}(\cup_{\check{x}<\tilde{x}}\tilde{\mu}_{\check{x}}))\cup(\cup_{\tilde{x}\geq\hat{x}}\tilde{\mu}_x)$ is at most one between agents of type x .

As $N_{x,\hat{x}}(\hat{\mu}) \geq N_{x,\hat{x}}(\mu)$ we know that $U(\hat{\mu}) \geq U(\mu)$. for x in descending order: suppose we break a link ii' where both agents are of type x; then i and i' are linked to agents j, j' respectively such that $x > \max(x_j, x_{j'})$ - but by supermodularity we know that every move will decrease aggregate utility as $Z(x,x) + Z(x_j, x_{j'}) > Z(x, x_j) + Z(x, x_{j'})$. As only perfect sorted networks can be efficient and thus strongly stable. It follows from Lemma 1 that only networks that are perfectly sorted can be pairwise stable. □

PROPOSITION 1: If there is supermodularity, a degree quota and constant decay with asymptotic independence ($\delta < (\kappa-1)^{-1}$) then there is asymptotic perfect sorting for strongly stable networks.

Proof. In order to describe sorting we need a concept of heterogeneity. Let *asymptotic heterogeneity* hold when the asymptotic share of at least two types is strictly positive share of the population.[18]

For the case of no externalities (i.e. δ=0) see the proof for Remark **??** below. In the remainder it is assumed that there are externalities $\delta \in (0,(\kappa-1)^{-1})$.

It is assumed that the asymptotic variance of x_i is strictly positive. This implies that there is a subset of types, $\hat{X} \subseteq X$, where $|\hat{X}| \geq 2$ and for every type $x \in \hat{X}$ it holds that there is an asymptotic strictly positive share of the total number of agents of that type, i.e., $\lim_{n\to\infty}(|\{i \in N_n\}_{x_i=x}|/n) > 0$. This follows as the number of types is finite $|X|$ is finite. Thus it is sufficient to simply analyze the network of agents who are of type $\hat{X}$. The argument is that for asymptotic infinite agents the measure of types not in $\hat{X}$ will have measure zero as no agent can have more than κ links.

Each agent will almost surely have κ links as it is assumed that each link adds positive value and there are asymptotic infinite agents (only a finite number can then not fulfill the degree quota).

Let the highest talented type with asymptotic support of be denoted $x = \max \hat{X}$. Let the second highest talented type in $\hat{X}$ be denoted $\tilde{x}$. Denote a given pair of agents of type x as i, i'. Fort his pair it is possible to compute: (i) an upper bound for agent i and i' of linking with two other agents of different types, and; (ii) a lower bound of forming a link between agent i and i'.

For deriving the lower bound of forming a link between agent i and i' we may use there are infinitely many agents of type x. As such it is possible that the neighbors of agent i and agent i' and all their neighbors and neighbors' neighbors etc. are linked with new agents of type x. Moreover, as there are infinitely many agents it is possible to assume that agent i and any of agent i's neighbors and neighbors' neighbors etc. have the shortest path to i' and any of agent i's neighbors and neighbors' neighbors via the link that agent i and i' forms. This implies that value of the link is equivalent to the sum of:

(i) The value of the link itself $Z(x,x)$.

(ii) The value for i and i' connecting with each others' neighbors and neighbors' neighbors etc. At each step l there are $(\kappa-1)^{\kappa}$ connections discounted at δ^{κ}. Note that both i and i' connect to each others' neighbors and neighbors' neighbors.

(iii) The value of indirectly connecting the neighbors and neighbors' neighbors etc. of agent i to the neighbors and neighbors' neighbors etc. of agent i' (calculated analogue to the previous step).

Thus the lower bound for agent i and i' of forming a link can be arbitrarily close to:

$$\begin{aligned}
& Z(x,x) \cdot \left\langle 1 + 2 \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l + 2 \cdot \sum_{q=1}^{\infty}\left[\langle(\kappa-1)\delta\rangle^q \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l\right]\right\rangle \\
= \quad & Z(x,x) + Z(x,x) \cdot 2 \cdot \frac{(\kappa-1)\delta}{1-(\kappa-1)\delta} + Z(x,x) \cdot 2 \cdot \left[\frac{(\kappa-1)\delta}{1-(\kappa-1)\delta}\right]^2 \qquad (12)
\end{aligned}$$

It is possible to compute the lower bound on the value of a link between two agents of type $\tilde{x}$ by the same procedure as above. When computing this it is assumed the agents of type $\tilde{x}$ has neighbors and neighbors' neighbors etc. that are all of type x. Thus the lower bound is

[18]Note the technical condition is that $\lim_{n\to\infty}(|\{i \in N_n\}_{x_i=\underline{x}}|/N) > 0$. That condition is satisfied when there is asymptotic variance of types, i.e. $\lim_{n\to\infty} \mathbb{V}_N[x_i] > 0$.

$$
\begin{aligned}
& Z(\tilde{x},\tilde{x}) + Z(x,\tilde{x}) \cdot 2 \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l + Z(x,x)\cdot 2 \cdot \sum_{q=1}^{\infty}\left[\langle(\kappa-1)\delta\rangle^q \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l\right] \\
= \quad & Z(\tilde{x},\tilde{x}) + Z(x,\tilde{x}) \cdot 2 \cdot \frac{(\kappa-1)\delta}{1-(\kappa-1)\delta} + Z(x,x)\cdot 2 \cdot \left[\frac{(\kappa-1)\delta}{1-(\kappa-1)\delta}\right]^2 \qquad (13)
\end{aligned}
$$

As there is monotonicity and supermodularity in link value the upper bound from linking with agents not of type x can be found by letting agent i and i' form a link each with respectively agents j and j' of type $\tilde{x}$ (second highest in $\hat{X}$). Let agents j and j' have neighborhood that corresponds to the neighborhoods of agent i and agent i' above. Let agent i and agent j (and same for agent i' and agent j') and all their neighbors and neighbors' neighbors etc. are linked with new agents of type x. Moreover, as there are infinitely many agents it is possible to assume that agent i and any of agent i's neighbors and neighbors' neighbors etc. have the shortest path to j and any of agent j's neighbors and neighbors' neighbors via the link that agent i and j forms. Thus it is possible to compute the upper bound of on the value of a link between i and j:

$$
\begin{aligned}
& Z(x,\tilde{x}) + \left\langle 1 + 2 \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l \right\rangle + Z(x,x)\cdot 2 \cdot \sum_{q=1}^{\infty}\left[\langle(\kappa-1)\delta\rangle^q \cdot \sum_{l=1}^{\infty}[(\kappa-1)\delta]^l\right] \\
= \quad & Z(x,\tilde{x}) + Z(x,\tilde{x}) \cdot 2 \cdot \frac{(\kappa-1)\delta}{1-(\kappa-1)\delta} + Z(x,x)\cdot 2 \cdot \left[\frac{(\kappa-1)\delta}{1-(\kappa-1)\delta}\right]^2 \qquad (14)
\end{aligned}
$$

By letting the lower bound on the value of the links ii', jj' (sum of Equation 12 and 13) be subtracted by the upper bound on the value of $ij, i'j'$ (twice the value of sum of Equation 14) to show that the difference is arbitrarily close:

$$
\begin{aligned}
& [Z(x,x) + Z(\tilde{x},\tilde{x}) - 2Z(x,\tilde{x})] + [Z(x,x) + Z(x,\tilde{x}) - 2Z(x,\tilde{x})] \cdot 2 \cdot \frac{(\kappa-1)\delta}{1-(\kappa-1)\delta} + Z(x,x)\cdot 2 \cdot \left[\frac{(\kappa-1)\delta}{1-(\kappa-1)\delta}\right]^2 \\
= \quad & [Z(x,x) + Z(\tilde{x},\tilde{x}) - 2Z(x,\tilde{x})] + [Z(x,x) - Z(x,\tilde{x})] \cdot 2 \cdot \frac{(\kappa-1)\delta}{1-(\kappa-1)\delta}
\end{aligned}
$$

As there is supermodularity and monotonicity we know the value of the difference in value in the equation above is positive. Thus at most finite agents of the highest type, x, will choose to partner with a lower type as the argument above can be repeated for an infinite number of agents of type x. So it holds that for agents of type x they will almost surely only partner with other agents of type x.

The argument can be repeated for type $\tilde{x}$ against any other type in $\hat{X}$. And the argument can be applied repeatedly until type $\min \hat{X}$ can only choose to link with other agents of type $\min \hat{X}$. □

C Local trees

This appendix treats the derivation of agents' benefits in a local tree network. Specifically when a link is deleted within a local tree and when it is formed across a network consisting of two components that are local trees.

Let $\Delta\#(n,\kappa) = \sum_{l=1}^{m(n,\kappa)}(\kappa \cdot (\kappa-1)^{l-1}) - n$. Let μ denote a local tree. We can express each agent's number of paths of length p as a function of number of agents and degree quota:

$$
\#_i^p(\mu) = \kappa(\kappa-1)^{p-1} - \mathbb{1}_{=m}(p) \cdot \Delta\#(n,\kappa), \qquad (15)
$$

where $\mathbb{1}_{=m}(p)$ is the Dirac measure of whether $p = m$. Note exact local trees are characterized by $\Delta\#(n,\kappa) = 0$. Let $m = m(n,\kappa)$ in this appendix section.

Suppose agents have homogeneous type x. We can express utility without transfers of each agent in the network μ as follows:

$$u_i(\mu) = \sum_{l=1}^{m} \#_i^l(\mu) \cdot \prod_{q=1}^{l} \delta^{r_q} \cdot z(x,x) \tag{16}$$

Exact local trees Let μ be a perfectly sorted network where the subset of links for each type is a component that can be classified as an exact local trees. Let networks μ_x and $\mu_{\tilde{x}}$ be the components associated with respectively types $x, \tilde{x} \in X$.

The essential property of exact local trees is between every pair of agents there is a unique shortest path of at most length m and the number of paths for every agent is prescribed by Equation 15. This can be deducted as follows. Note first that the fact that the number of walks with at most length m starting in a given agent i cannot exceed $\sum_{p=1}^{m} \#_i^p(\cdot)$. Recall also that local trees has the property that all agents are reached within distance m. Moreover exact local trees has the property that for any agent i it holds that $n-1 = \sum_{p=1}^{m} \#_i^p(\mu)$; thus all shortest paths with distances less than or equal to m must be unique path between the two particular agent which is at most length m.

The uniqueness and countability of paths can be used to infer the losses when links are either removed or added to an exact local tree.

Exact local trees - loss from deletion Suppose two links $\iota\iota', jj' \in \mu$ are deleted. Suppose also agents ι and j have respectively type x and $\tilde{x}$ and thus the two links are not from the same component. Let the new network that results from removal of the links be denoted $\hat{\mu} = \mu \backslash \{\iota\iota', jj'\}$. This notation is used throughout out. It is sufficient to analyze what happens to the network component with agents of type $x =$ as the other component can be analyzed by repeating the analysis for type x.

Let i denote a generic agent of type x. Let the shortest path in μ from i to either ι or ι' be denoted $\hat{p}_i$ where $\hat{p}_i = \min(p_{i\iota}(\hat{\mu}), p_{i\iota'}(\hat{\mu}))$. When $\hat{p}_i = 0$ then either $i = \iota$ or $i = \iota'$.

The deletion of link $\iota\iota'$ implies that any pair of agents i, i' whose unique shortest path in μ includes the link $\iota\iota'$ will have a new shortest path which exceeds m. For exact local trees we can exactly determine the length of the new path.

When link $\iota\iota'$ is deleted we can show there is no shortest path between i and i' in $\tilde{\mu}$ with length below $2m - \hat{p}_i - \hat{p}_{i'}$. We show this by demonstrating there is no agent j such that $p_{ji}(\tilde{\mu}) + p_{ji'}(\tilde{\mu}) < 2m - \hat{p}_i - \hat{p}_{i'}$. This statement can be shown as follows. An implication of the statement is $p_{j\iota}(\tilde{\mu}) + p_{j\iota'}(\tilde{\mu}) < 2m$, however, we can show this derivative statement is impossible. If $p_{j\iota}(\tilde{\mu}) + p_{j\iota'}(\tilde{\mu}) < 2m$ then agent j is such that either ($p_{j\iota}(\mu) < m$ and $p_{j\iota'}(\mu) \leq m$) or ($p_{j\iota'}(\mu) \leq m$ and $p_{j\iota}(\mu) < m$) which means there are two paths with length $\leq m$ which violates the structure of exact local trees; the paths are respectively either between j and ι' or between j and ι.

We can now show that when link $\iota\iota'$ is deleted the new shortest path between i and i' in $\tilde{\mu}$ has a length of exactly $2m - \hat{p}_i - \hat{p}_{i'}$. This is shown by demonstrating there is an agent j such that $p_{ji}(\tilde{\mu}) = m - \hat{p}_i$ and $p_{ji'}(\tilde{\mu}) = m - \hat{p}_{i'}$. This can be shown follows. Suppose that $p_{ji}(\tilde{\mu}) = m - \hat{p}_i$. We will demonstrate that $p_{ji'}(\tilde{\mu}) = m - \hat{p}_{i'}$. As $p_{ji}(\tilde{\mu}) = m - \hat{p}_i$ it follows that $p_{j\iota}(\tilde{\mu}) = m$. From the definition of exact local trees there must exist a path of length less than m between j and ι' in network μ. As the previous paragraph demonstrated that neither of these paths can be strictly shorter than m they must both be exactly m.

The number of shortest paths of length p which become altered for agent i is $(\kappa-1)^{p-\hat{p}_i-1}$ for $p = \hat{p}_i, .., m-2, m-1$. This can be demonstrated as follows. If agent $p_{i\iota}(\mu) = m$ and $p_{i\iota'}(\mu) = m$ then no shortest paths are altered; this is clear as agent i as none of the unique shortest paths includes $\iota\iota'$ as they have at most length m. If instead $p_{i\iota}(\mu) = m-1$ then the unique shortest path from i to ι' includes $\iota\iota'$ is the last link; this implies a new shortest path if $\iota\iota'$ is deleted. Thus if $p_{i\iota}(\mu) = m-1$

then one shortest path of length m is lost. When $p_{i\iota}(\mu) = m-2$ then one path of length $m-1$ is lost by the same argument; moreover $\kappa - 1$ paths that has $\iota\iota'$ as the second last link. By induction this can be done at higher order and thus for shorter distances.

Using the number of rerouted paths shown above we can establish the total number of shortest paths in network $\tilde{\mu}$ for agent i that has a length of p:

$$\#_i^p(\hat{\mu}) = \begin{cases} \kappa(\kappa-1)^{p-1} - \mathbb{1}_{>\hat{p}_i}(p) \cdot (\kappa-1)^{p-\hat{p}_i-1}, & p \leq m \\ (\kappa-1)^{2m-\hat{p}_i-p}, & p \in (m, 2m-\hat{p}_i]. \end{cases} \tag{17}$$

By combining the count of shortest paths rerouted with their new length we can generalize the loss for any agent from the deletion of link $\iota\iota'$ when all agents are homogeneous of type x:

$$u_i(\mu) - u_i(\hat{\mu}) = \sum_{l=1}^{m-\hat{p}_i} \left[(\kappa-1)^{l-1} \cdot \left(\prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} - \prod_{q=1}^{2m-(l-1)-\hat{p}_i} \delta^{r_q} \right) \right] \cdot z(x,x) \tag{18}$$

We can aggregate the losses across homogeneous agents and we arrive at the following expression:

$$U(\mu) - U(\hat{\mu}) = \sum_{l=1}^{m} \left[2l \cdot (\kappa-1)^{l-1} \cdot \left(\prod_{q=1}^{l-1} \delta^{r_q} - \prod_{q=1}^{2m-(l-1)} \delta^{r_q} \right) \right] \cdot z(x,x) \tag{19}$$

Exact local trees - gains from linking across types Having made the losses explicit in exact local trees we move on to establishing the gains of establishing a link in a perfectly sorted network.

Suppose now that agents ι and j have respectively type x and $\tilde{x}$; let these two agents form a link across types. In order to link they both need to sever a link each - denote the link being deleted as respectively $\iota\iota'$ and jj'. The costs associated with this deletion have already been calculated above.

The gains to agents ι and j of forming a link ιj are direct benefits and the new indirect connections that are accessed through the link ιj. For agent ι the benefits from forming a link with j can be computed with Equation 17 where the input length is added one (as ιj is added to the shortest path). Let $\breve{\mu} = \{\iota j\} \cup \mu$.

$$u_\iota(\breve{\mu}) - u_\iota(\mu) = \left[\sum_{l=1}^{m} (\kappa-1)^{l-1} \cdot \prod_{q=1}^{l-1} \delta^{r_q} + \sum_{l=1}^{m-1} (\kappa-1)^{l-1} \cdot \prod_{q=1}^{2m-(l-1)} \delta^{r_q} \right] \cdot z(x, \tilde{x}).$$

The above expression is relevant for evaluating the pairwise gains as it captures individual benefits for a pairwise formation of a link by ι and j. However, we are also interested in the sub-connected network as it allows to assess the efficiency. Suppose instead now that ι' and j' also form a link; thus $\iota j, \iota' j'$ are formed while $\iota\iota', jj'$ are deleted. Let $\tilde{\mu} = \mu \cup \{\iota j, \iota' j'\} \backslash \{\iota\iota', jj'\}$.

Let i be an agent of type x and let $\hat{p}_i$ still denote the least distance to either ι or ι'. We can calculate the benefits for i when $\iota j, \iota' j'$ are formed. The benefits are the indirect connections to agents of type $\tilde{x}$ with whom agent i has no connections in μ. Again the number of new connections can be computed for a given path length. We can use that i and i' of type $\tilde{x}$ can be at most $2m+1$ away from each other. This follows from the fact that $p_{i\iota} + p_{i\iota'} = 2m$ and $p_{i'j} + p_{i'j'} = 2m$. Thus it must be that $p_{ij} + p_{ij'} = 2m+2$ and therefore

$$p_{ii'} = \min(p_{ij} + p_{i'j}, p_{ij'} + p_{i'j'}) = \min(p_{ij} + p_{i'j}, 4m + 2 - p_{ij} - p_{i'j}) \tag{20}$$

From the above expression it follows that $p_{ii'} \leq 2m+1$ as the expression is maximized for $p_{ij} + p_{i'j} = 2m+1$.

The number of shortest paths from i through ιj to agents of the other type $\tilde{x}$ can be found using Equation 17 for agent ι and using the added extra distance $1+\hat{p}_i$:[19] for distance $p \in \{1+\hat{p}_i, ..., m+1+\hat{p}_i\}$ there are $(\kappa-1)^{p-1-\hat{p}_i}$ agents; $p \in \{m+2+\hat{p}_i, 2m+1\}$ there are $(\kappa-1)^{2m+1-(p-1-\hat{p}_i)}$. The shortest paths from i not routed through ι but instead through ι' are those where $p+1+\hat{p}_i > 2m+1$; from Equation 20 we know the new shortest path length is $4m+2-p-1-\hat{p}_i$. The number of shortest paths through ι' in network $\tilde{\mu}$ will be $(\kappa-1)^{2m+1-(p-1-\hat{p}_i)}$ and the new length $4m+2-p-1-\hat{p}_i$. Putting these facts together we have:

$$\#_i^p(\tilde{\mu}) - \#_i^p(\hat{\mu}) = \begin{cases} (\kappa-1)^{p-1-\hat{p}_i}, & p \in \{\hat{p}_i+1, .., m+1+\hat{p}_i\}, \\ (\kappa-1)^{2m+1-p-\hat{p}_i}, & p \in \{m+\hat{p}_i+2, .., 2m+1\}, \\ (\kappa-1)^{p+\hat{p}_i-2m-1}, & p \in \{2m+1-\hat{p}_i, .., 2m\}. \end{cases} \tag{21}$$

From the number of paths above we can derive the change in utility from when $\iota j, \iota' j'$ are added to the network for a given agent i of type x.

$$u_i(\tilde{\mu}) - u_i(\hat{\mu}) = \begin{bmatrix} & \sum_{l=0}^{m} (\kappa-1)^l \cdot \delta^{l+\hat{p}_i} \\ + & \sum_{l=\hat{p}_i}^{m-1} (\kappa-1)^l \cdot \delta^{2m-l+\hat{p}_i} \\ + & \sum_{l=0}^{\hat{p}_i-1} (\kappa-1)^l \cdot \delta^{2m+l-\hat{p}_i} \end{bmatrix} \cdot z(x, \tilde{x}) \tag{22}$$

By aggregating over all agents of type the gain in benefits by forming $\iota j, \iota' j'$ is as follows:

$$U(\tilde{\mu}) - U(\hat{\mu}) = \sum_{p=0}^{m} \left(\begin{bmatrix} \mathbb{1}_{<m}(p) \cdot 2 \cdot (\kappa-1)^p + \\ \mathbb{1}_{=m}(p) \cdot (n - 2 \cdot \sum_{l=1}^{m-1} (\kappa-1)^l) \end{bmatrix} \cdot \begin{bmatrix} & \sum_{l=0}^{m} (\kappa-1)^l \cdot \delta^{l+p} \\ + & \sum_{l=p}^{m-1} (\kappa-1)^l \cdot \delta^{2m-l+p} \\ + & \sum_{l=0}^{p-1} (\kappa-1)^l \cdot \delta^{2m+l-p} \end{bmatrix} \right) \cdot Z(x, \tilde{x}). \tag{23}$$

Local trees We can use the analysis above on exact local trees to bound the gains and losses for (non-exact) local trees. Recall that exact local trees has the property that $\Delta\#(n,\kappa) = 0$ and for while for non-exact local trees $\Delta\#(n,\kappa) > 0$; thus the only difference between exact and non-exact local trees is that for a given agent the number of connected other agents at exactly distance m is lower for non-exact local trees.

Using the analysis of exact local trees we can compute the bounds on loss of utility for a given agent in the local when a link is deleted - this is done by reusing Equation 17 as follows. We can discount the number of agents initially at distance m by $\Delta\#$. Moreover, the new distance between agents i and i' after deletion of the link $\iota\iota'$ is at least $\min(p_{ii'}, 2m-2-\hat{p}_i-\hat{p}_{i'})$ at most $2m-\hat{p}_i-\hat{p}_{i'}$.[20] From these two facts we can derive the bound on loss of utility when $\iota\iota'$ is deleted. The upper bound on loss (in terms of magnitude) is when new shortest paths have most distance (i.e. $2m-\hat{p}_i-\hat{p}_{i'}$); the lower bound is found when new distance is least (i.e. $\min(p_{ii'}, 2m-2-\hat{p}_i-\hat{p}_{i'})$):

$$u_i(\mu) - u_i(\hat{\mu}) \leq \sum_{l=1}^{m-\hat{p}_i} \left[\max(0, (\kappa-1)^{l-1} - \mathbb{1}_{=m}(l) \cdot \Delta\#(n,\kappa)) \left(\prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} - \prod_{q=1}^{2m-(l-1)-\hat{p}_i} \delta^{r_q} \right) \right] \cdot z(x,x), \tag{24}$$

$$u_i(\mu) - u_i(\hat{\mu}) \geq \sum_{l=1}^{\tilde{m}} \left[(\kappa-1)^{l-1} \cdot \left(\prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} - \prod_{q=1}^{2m-(l+1)-\hat{p}_i} \delta^{r_q} \right) \right] \cdot z(x,x), \quad \tilde{m} = \min(m-1, m-\hat{p}_i). \tag{25}$$

[19]The replacement has to hold as the shortest paths from i must both include ιj and the shortest path from i to j.

[20]The upper bound follows from the fact that for any two agents i and i' in the local tree there is still always an agent j at distances $p_{ij} = m - \hat{p}_i$ and $p_{i'j} = m - \hat{p}_{i'}$. The lower bound can be established by repeating an argument used for exact local trees. If the new distance between two agents i and i' after deletion of $\iota\iota'$ had been less than $\min(p_{ii'}(\mu), 2m-2-\hat{p}_i-\hat{p}_{i'})$ then the following would be true. There would be multiple shortest paths of length less than or equal to $m-1$ between either (ι and j) or (ι' and j). This would violate the property of local trees that all shortest paths of length $\leq m-1$ are unique.

Fact 2. For the perfectly sorted network μ which consists of $|X|$ network components which each constitute a local tree with $n/|X|$ agents, then it holds that for any agent i is of type x and $\hat{p}_i > 0$:

$$u_i(\hat{\mu}) - u_i(\mu) > \prod_{l=1}^{\hat{p}_i} \delta^{r_l} \cdot [u_\iota(\hat{\mu}) - u_\iota(\mu)] \tag{26}$$

Proof. Inequality 26 can be rewritten into: $\prod_{l=1}^{\hat{p}_i} \delta^{r_l} \cdot [u_\iota(\mu) - u_\iota(\hat{\mu})] > u_i(\mu) - u_i(\hat{\mu})$. A sufficient condition for this rewritten expression is $u_\iota(\mu) - u_\iota(\hat{\mu}) \geq u_i(\mu) - u_i(\hat{\mu})$ as both $u_\iota(\mu) - u_\iota(\hat{\mu}) \geq 0$ and $u_i(\mu) - u_i(\hat{\mu}) \geq 0$ as well as $\prod_{l=1}^{\hat{p}_i} \delta^{r_l} \in [0,1)$. This new sufficient condition will hold if the expression below is valid (the expression can be derived by substituting in Inequality 25 for agent ι and Inequality 24 for agent i):

$$\sum_{l=1}^{m-1} \left[(\kappa-1)^{l-1} \cdot \left(\prod_{q=1}^{l-1} \delta^{r_q} - \prod_{q=1}^{2m-(l+1)} \delta^{r_q} \right) \right] \geq \sum_{l=1}^{m-\hat{p}_i} \left[(\kappa-1)^{l-1} \left(\prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} - \prod_{q=1}^{2m-(l-1)-\hat{p}_i} \delta^{r_q} \right) \right]$$

The above inequality holds if for every $l = 1, ..., m-1$ the expression below is satisfied for $\hat{p}_i > 0$:

$$\begin{aligned}
\prod_{q=1}^{l-1} \delta^{r_q} - \prod_{q=1} \delta^{r_q} &\geq \prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} - \prod_{q=1}^{2m-(l-1)-\hat{p}_i} \delta^{r_q} \\
\left(\prod_{q=1}^{l-1} \delta^{r_q} \right) \left[1 - \prod_{q=l}^{2m-1-l} \delta^{r_q} \right] &\geq \left(\prod_{q=1}^{l-1+\hat{p}_i} \delta^{r_q} \right) \left[1 - \prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q} \right] \\
\left[1 - \prod_{q=l}^{2m-1-l} \delta^{r_q} \right] &\geq \left(\prod_{q=l}^{l-1+\hat{p}_i} \delta^{r_q} \right) \left[1 - \prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q} \right]
\end{aligned}$$

As $\left(\prod_{q=l}^{l-1+\hat{p}_i} \delta^{r_q} \right) \leq 1$ it remains to be shown that $1 - \prod_{q=l}^{2m-1-l} \delta^{r_q} \geq 1 - \prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q}$. This is equivalent to showing that $\prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q} \geq \prod_{q=l}^{2m-1-l} \delta^{r_q}$. As for any $q \in \mathbb{N}$: $r_{q+1} \leq r_q$ and $r_q \in [0,1]$ it follows that $\delta^{r_{q+1}} \geq \delta^{r_q}$. Thus it must be that $\prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q} \geq \prod_{q=l}^{2m+1-l-2\hat{p}_i} \delta^{r_q}$. Therefore, as it holds that $\prod_{q=l}^{2m-1-l} \delta^{r_q} \leq \prod_{q=l}^{2m+1-l-2\hat{p}_i} \delta^{r_q}$ then $\prod_{q=l}^{2m-1-l} \delta^{r_q} \leq \prod_{q=l+\hat{p}_i}^{2m+1-l-\hat{p}_i} \delta^{r_q}$. □

We can also derive bounds on the gains from connecting across types for local trees. We will not do this explicitly but instead use definition 9 on symmetric losses in local trees. This allows to express our next result:

Fact 3. For the perfectly sorted network μ which consists of $|X|$ network components which each constitute a local tree of $n/|X|$ agents that has symmetric losses then it holds that for agents i, ι of type x and $\hat{p}_i > 0$

$$u_i(\tilde{\mu}) - u_i(\hat{\mu}) \geq \prod_{l=1}^{\hat{p}_i} \delta^{r_l} \cdot [u_\iota(\breve{\mu}) - u_\iota(\hat{\mu})], \tag{27}$$

where $\hat{\mu} = \mu \backslash \{\iota\iota', jj'\}$ and $\tilde{\mu} = \hat{\mu} \cup \{\iota j, \iota' j'\}$ as well as $\breve{\mu} = \hat{\mu} \cup \{\iota j\}$.

Proof. It holds that $u_\iota(\tilde{\mu}) - u_\iota(\hat{\mu}) \geq u_\iota(\breve{\mu}) - u_\iota(\hat{\mu})$ as $\tilde{\mu} \subseteq \breve{\mu}$ (thus all shortest paths in $\tilde{\mu}$ cannot have a length that exceeds that in $\breve{\mu}$). Therefore it suffices to show:

$$u_i(\tilde{\mu}) - u_i(\hat{\mu}) \geq \prod_{l=1}^{\hat{p}_i} \delta^{r_l} \cdot [u_\iota(\tilde{\mu}) - u_\iota(\hat{\mu})]. \tag{28}$$

As the local tree has symmetric losses it follows that $u_\iota(\tilde{\mu}) - u_\iota(\hat{\mu}) = u_{\iota'}(\tilde{\mu}) - u_{\iota'}(\hat{\mu})$; this follows from the fact that they both gain an equal number of new shortest paths through j, j', this follows as as j, j' have same number of paths after deletion of jj' due to symmetric losses. This entails that without loss of generality we can assume that $p_{i\iota} = \hat{p}_i$ as otherwise we could substitute ι with ι' and conduct the analysis again.

For ι and some agent i' of type $\tilde{x}$ it holds that $p_{ii'}(\tilde{\mu}) \leq p_{\iota i'}(\tilde{\mu}) + \hat{p}_i$. This follows as there exists a path between i, ι and ι, i' with respectively lengths $p_{\iota i'}(\tilde{\mu})$ and $\hat{p}_i$; thus $p_{ii'}(\tilde{\mu}) \leq p_{\iota i'}(\tilde{\mu}) + \hat{p}_i$. This implies the following inequality must hold:

$$\sum_{x_{i'}=\tilde{x}} \left(\prod_{l=1}^{p_{ii'}(\tilde{\mu})} \delta^{r_l} \right) \geq \left(\prod_{l=1}^{p_{\iota i}(\tilde{\mu})} \delta^{r_l} \right) \cdot \sum_{x_{i'}=\tilde{x}} \left(\prod_{l=1}^{p_{\iota i'}(\tilde{\mu})} \delta^{r_l} \right)$$

As $u_\iota(\tilde{\mu}) - u_\iota(\hat{\mu}) = \sum_{x_i=\tilde{x}} \prod_{l=1}^{p_{\iota i'}(\tilde{\mu})} \delta^{r_l} \cdot z(x, \tilde{x})$ and $u_i(\tilde{\mu}) - u_i(\hat{\mu}) = \sum_{x_i=\tilde{x}} \prod_{l=1}^{p_{ii'}(\tilde{\mu})} \delta^{r_l} \cdot z(x, \tilde{x})$ it follows that Inequality 28 holds which proves our fact.

□

D Over-sorting

Proposition 2: Suppose there is supermodularity, hyperbolic decay and a degree quota, it follows that for any network μ in $M^{sort:comp}$ if μ satisfies both a binding degree quota (i.e. $\forall i : k_i(\mu) = \kappa$) and having each component $\tilde{\mu} \subseteq \mu$ being cyclic then μ has over-sorting as follows:

$$\underline{\delta} \leq \min_{x,\tilde{x} \in X} \left(\frac{\hat{Z}_{x,\tilde{x}} - 1}{\hat{Z}_{x,\tilde{x}} - 1 + \frac{1}{2} n_x \cdot n_{\tilde{x}}} \right) \text{ and } \bar{\delta} \geq \min_{x,\tilde{x} \in X} \left(\frac{\hat{Z}_{x,\tilde{x}} - 1}{\max(n_x, n_{\tilde{x}}) + \hat{Z}_{x,\tilde{x}} - 1} \right), \quad \hat{Z}_{x,\tilde{x}} = \frac{Z(x,x)+Z(\tilde{x},\tilde{x})}{2Z(x,\tilde{x})}$$

Proof. As the network is cyclic when a link is deleted any two agents μ_x are still connected. Thus although shortest paths may be shorter, only two agents who had the link will have a change in path length from one to something higher. Thus when a link ii' is deleted in μ_x the loss for agent i, i' is $(1 - \delta) \cdot z(x, x)$ while no other agents incures a loss.

Suppose two agents i, j of distinct types respectively $x, \tilde{x}$ deviate by forming a link and delete a link each from μ. The total loss for i and j for deleting a link each is:

$$\begin{aligned} & (1 - \delta) \cdot [z(x, x) + z(\tilde{x}, \tilde{x})] \\ = \; & (1 - \delta) \cdot \hat{Z}_{x,\tilde{x}} \cdot Z(x, \tilde{x}) \end{aligned}$$

The benefits gained for agent i for establishing a link to j is $[1 + (n_{\tilde{x}} - 1) \cdot (1 - \delta)] \cdot z(x, \tilde{x})$. Thus the total benefits gained for i and j from pairwise deviation can be bounded as follows.

$$\begin{aligned} & [1 + (n_x - 1) \cdot \delta] \cdot z(x, \tilde{x}) + [1 + (n_{\tilde{x}} - 1) \cdot \delta] \cdot z(\tilde{x}, x) \\ \leq \; & [1 + (\max(n_x, n_{\tilde{x}}) - 1) \cdot \delta] \cdot Z(x, \tilde{x}) \end{aligned}$$

We can check when pairwise stability by using the upper bound for gains in benefits from above:

$$\begin{aligned} (1 - \delta) \cdot \hat{Z}_{x,\tilde{x}} \cdot Z(x, \tilde{x}) &\geq [1 + (\max(n_x, n_{\tilde{x}}) - 1) \cdot \delta] \cdot Z(x, \tilde{x}) \\ (1 - \delta) \cdot \hat{Z}_{x,\tilde{x}} &\geq 1 + 1 \cdot (\max(n_x, n_{\tilde{x}}) - 1) \cdot \delta \\ \hat{Z}_{x,\tilde{x}} - 1 &\geq [1 \cdot (\max(n_x, n_{\tilde{x}}) - 1) + \hat{Z}_{x,\tilde{x}}] \cdot \delta \\ \frac{\hat{Z}_{x,\tilde{x}} - 1}{\max(n_x, n_{\tilde{x}}) + \hat{Z}_{x,\tilde{x}} - 1} &\geq \delta \qquad (29) \end{aligned}$$

Thus we have shown that for any two types pairwise stability holds when Inequality 29 is satisfied. Thus we can establish a lower bound for $\bar{\delta}$ (i.e. the upper bound in δ for pairwise stability of μ) by taking the minimum of left-hand-side in Inequality 29; thus it follows that: $\bar{\delta} \geq \min_{x,\tilde{x}\in X}\left(\frac{\hat{Z}_{x,\tilde{x}}-1}{\max(n_x,n_{\tilde{x}})+\hat{Z}_{x,\tilde{x}}-1}\right)$.

We can do a similar procedure for finding $\underline{\delta}$ for when the aggregate gains of establishing links across types is zero. We will evaluate the sub-connected network $\tilde{\mu} = \mu \cup \{ij, i'j'\}\backslash\{ii', jj'\}$. The total loss $U(\mu) - U(\mu\backslash\{ii', jj'\})$ is equal to double that of i, j suffers, i.e. $2(1-\delta)\cdot\hat{Z}_{x,\tilde{x}}\cdot Z(x,\tilde{x})$.

The gains from connecting are two links of value $Z(x,\tilde{x})$ as well as $n_x \cdot n_x - 2$ indirect connections (between all agents of type x and $\tilde{x}$) of value $Z(x,\tilde{x})\cdot\delta$. It follows that the total gains in benefits are $Z(x,\tilde{x})\cdot[2 + (n_x\cdot n_x - 2)\cdot\delta]$. The threshold $\underline{\delta}$ can be found from finding when gains equal losses:

$$\begin{aligned}
2(1-\delta)\cdot\hat{Z}_{x,\tilde{x}}\cdot Z(x,\tilde{x}) &= 2Z(x,\tilde{x}) + (n_x\cdot n_x - 2)\cdot Z(x,\tilde{x})\cdot\delta \\
(1-\delta)\cdot\hat{Z}_{x,\tilde{x}} &= 1 + (\frac{1}{2}n_x\cdot n_x - 1)\cdot\delta \\
\frac{\hat{Z}_{x,\tilde{x}} - 1}{\hat{Z}_{x,\tilde{x}} - 1 + \frac{1}{2}n_x\cdot n_x} &= \delta
\end{aligned} \tag{30}$$

It follows that the threshold $\underline{\delta}$ such that every δ above implies there exists an efficient deviation from μ must be at least the minimum of left-hand-side in Equation 30 over possible types, i.e. it must hold that $\underline{\delta} \leq \min_{x,\tilde{x}\in X}\frac{\hat{Z}_{x,\tilde{x}}-1}{\hat{Z}_{x,\tilde{x}}-1+\frac{1}{2}n_x\cdot n_x}$. □

Theorem 2: When there is supermodularity, a degree quota κ and each type has identical number of agents then networks segregated into local trees with symmetric losses are over-sorted.

Proof. Let μ be a network which is segregated into $|X|$ components where each component is a local tree with $n/|X|$ agents. Let there be no transfers between any agents.

As each subnetwork for a given type is a local tree it is stable against deviations by agents of the same type - this follows as local trees provides maximal possible benefits in the subnetwork for all its agents. Thus only two agents of different types may have a feasible deviation.

Let ι, j be agents of respectively types x and $\tilde{x}$. These two agents can deviate by each deleting a link to ι' and j' respectively while jointly forming a link - the new network resulting from this move is denoted $\hat{\mu} = \tilde{\mu}\backslash\{\iota\iota'\}$. An alternative network is $\tilde{\mu}$ where the links $\iota\iota', jj'$ are removed while the links $\iota j, \iota', j'$ have been formed; thus $\tilde{\mu} = \mu \cup \{\iota j, \iota' j'\}\backslash\{\iota\iota', jj'\}$.

Define the gross loss of benefits for i as $u_i(\hat{\mu}) - u_i(\mu)$ while the gross gains are $u_i(\tilde{\mu}) - u_i(\hat{\mu})$. There must exist a threshold of externalities $\bar{\delta} \in (0,1)$ where μ is no longer pairwise stable as cost of deviation monotonically decreases and approaches zero as $\delta \to 1$ while gains are monotonically increasing. The monotonicity of losses follows as the gross loss consists of the shortest paths that includes $\iota\iota'$ may be shorter in $\hat{\mu}$; thus the gross loss is mitigated by a higher δ as longer shortest paths are punished less. The monotonicity of gains follows as the gains consist of new shortest paths to agents of type $\tilde{x}$ through ιj and $j'\iota'$ as gains from new indirect connections are higher.

From Fact 2 and 3 in Appendix C it follows that:

$$u_i(\tilde{\mu}) - u_i(\mu) > \delta^{\min(p_{i\iota}(\tilde{\mu}), p_{i\iota'}(\tilde{\mu}))}[u_\iota(\tilde{\mu}) - u_\iota(\mu)].$$

Aggregating for all agents this implies:

$$U(\tilde{\mu}) - U(\tilde{\mu}) \quad > \quad \sum_{x_i = x} \left(\delta^{\min(p_{i\iota}(\tilde{\mu}), p_{i\iota'}(\tilde{\mu}))} [u_\iota(\tilde{\mu}) - u_\iota(\mu)] \right) + \sum_{x_i = \tilde{x}} \left(\delta^{\min(p_{ij}(\mu), p_{ij'}(\mu))} [u_j(\tilde{\mu}) - u_j(\mu)] \right)$$

$$U(\tilde{\mu}) - U(\tilde{\mu}) \quad > \quad \left[2 \cdot \sum_{l=0}^{m-1} \left((\kappa - 1)^l \cdot \delta^l \right) + \left(n - 2 \cdot \sum_{l=0}^{m-1} (\kappa - 1)^l \right) \cdot \delta^m \right] \cdot [u_\iota(\tilde{\mu}) - u_\iota(\mu) + u_j(\tilde{\mu}) - u_j(\mu)],$$

where $m = m(n, \kappa)$. The inequality above implies the following: if $U(\tilde{\mu}) - U(\mu) = 0$ then $u_\iota(\tilde{\mu}) - u_\iota(\mu) + u_j(\tilde{\mu}) - u_j(\mu) < 0$; $U(\tilde{\mu}) - U(\mu) > 0$ when $u_\iota(\tilde{\mu}) - u_\iota(\mu) + u_j(\tilde{\mu}) - u_j(\mu) = 0$.

It can be argued that that there must exist a threshold, $\underline{\delta}$, exists such that for $\delta = \underline{\delta}$ then $U(\tilde{\mu}) - U(\mu) = 0$ and that $\underline{\delta} < \bar{\delta}$. This follows as $U(\tilde{\mu}) - U(\mu) < 0$ for $\delta = 0$ and $U(\tilde{\mu}) - U(\mu) > 0$ when $u_\iota(\tilde{\mu}) - u_\iota(\mu) + u_j(\tilde{\mu}) - u_j(\mu) = 0$ as well as continuity of $U(\tilde{\mu}) - U(\mu)$ in δ.

□

E Monotonic centrality

THEOREM 3: Suppose there are no externalities and monotonicity in link value then the set of pairwise stable networks has degree monotonicity.

Proof. Suppose the claim is false; that is, for some pairwise stable network $\mu \in M^{ps}$ it holds for two agents i and j that $x_i > x_j$ but $k_i(\mu) < k_j(\mu)$. The condition that $k_i(\mu) < k_j(\mu)$ entails there is another agent in j's network which is neither in i's network nor is agent i, i.e. $\nu_j(\mu) \backslash (\nu_i(\mu) \cup \{i\}) \neq \emptyset$.

Let $j' \in \nu_j(\mu) \backslash (\nu_i(\mu) \cup \{i\})$. From monotonicity in link value it holds that $Z_{ij'} > Z_{jj'}$ as $x_i > x_j$. Moreover, from the cost technologies (either convex or a degree quota) it must be that $c_i(\mu \cup ij') - c_i(\mu) \leq c_j(\mu) - c_j(\mu \backslash jj')$ as $k_i(\mu) < k_j(\mu)$. These two facts from benefits and costs entail that the value created by forming ij' and deleting jj' will be equivalent to:

$$Z_{ij'} - (c_i(\mu \cup ij') - c_i(\mu)) > Z_{jj'} - (c_j(\mu) - c_j(\mu \backslash jj'))$$

As the move to $(\mu \cup ij') \backslash jj'$ is feasible (and respects the degree quota if there is one), it follows that strong stability is violated as it implies that μ is not efficient. From Lemma 1 it follows that pairwise stability is also violated if the claim is false. □

PROPOSITION 4: Suppose there are externalities as well as monotonicity and no modularity then the set of strongly stable networks has δ-decay monotonicity.

Suppose there is monotonicity and no modularity in link value then for every $\mu \in M^{ss}$ it holds that the decay centrality of an agent i in μ, $d_i^\delta(\mu)$, is (weakly) monotone increasing in type:

$$\forall \mu \in M^{ss}: \ x_i > x_j \Rightarrow d_i^\delta(\mu) \geq d_j^\delta(\mu).$$

Proof. The no modularity condition implies that individual link value is independent of own talent and thus separable for any $\hat{x} \in X$: $Z(\hat{x}, \tilde{x}) = \tilde{Z}(\tilde{x}) + \tilde{Z}(\hat{x})$ where $\tilde{Z}$ is the contribution to the link value for a given level of talent. This entails that we can rewrite the aggregate benefits using Equation 1:

$$\sum_{i \in N} b_i(\mu) = \sum_{i \in N} \sum_{j \neq i} \delta^{p_{ij}(\mu) - 1} z_{i,j} = \sum_{ij \in \mu^c} \delta^{p_{ij}(\mu) - 1} Zi, j = \sum_{i \in N} \sum_{j \neq i} \delta^{p_{ij}(\mu) - 1} \tilde{Z}(x_i) = \sum_{i \in N} d_i^\delta(\mu) \cdot \tilde{Z}(x_i)$$

Due to monotonicity in link value it also holds that $\frac{\partial}{\partial x_i} Z(x_i, x_{i'}) = \frac{\partial}{\partial x_i} \tilde{Z}(x_{i'}) > 0$. This entails that a necessary condition for the sum of utilities to be maximal by some network μ is that for any two agents i', j' such that $x_{i'} > x_{j'}$ it holds that $d_{i'}^\delta(\mu) > d_j^\delta(\mu)$. The necessity is demonstrated in the following.

Denote an alternative network $\tilde{\mu}$ where i' and j' have switched positions: if i' and j' are not linked in μ then let $\nu_i(\tilde{\mu}) = \nu_{i'}(\mu)$ and $\nu_{i'}(\tilde{\mu}) = \nu_i(\mu)$; else if i' and j' are linked in μ then let $\nu_i(\tilde{\mu}) = \nu_{i'}(\mu) \cup \{i'\}/\{i\}$ and $\nu_{i'}(\tilde{\mu}) = \nu_i(\mu) \cup \{i\}/\{i'\}$. Thus it holds that $d_j^\delta(\tilde{\mu}) = d_i^\delta(\mu)$ and $d_i^\delta(\tilde{\mu}) = d_j^\delta(\mu)$ A deviation from μ to $\tilde{\mu}$ is possible for the grand coalition.

We will show that the alternative network $\tilde{\mu}$ will generate higher aggregate utility which violates efficiency of μ and thus the strong stability. Starting with costs there are two cases of cost technology: when there is quota in links then a deviation to the alternative network $\tilde{\mu}$ is consistent with the degree quota[21] and has unchanged costs; when there are convex costs then the move will have no change in aggregate costs as the sum of costs for agents i and j is unchanged. However, the benefits will be higher under $\tilde{\mu}$, using that i and i switch neighborhoods;

$$
\begin{aligned}
\sum_{i \in N} b_i(\tilde{\mu}) &> \sum_{i \in N} b_i(\mu) \\
d_i^\delta(\tilde{\mu}) \cdot \tilde{Z}(x_i) + d_j^\delta(\tilde{\mu}) \cdot \tilde{Z}(x_j) &> d_i^\delta(\mu) \cdot \tilde{Z}(x_i) + d_j^\delta(\mu) \cdot \tilde{Z}(x_j) \\
d_j^\delta(\mu) \cdot [\tilde{Z}(x_i) - \tilde{Z}(x_j)] &> d_i^\delta(\mu) \cdot [\tilde{Z}(x_i) - \tilde{Z}(x_j)] \\
d_j^\delta(\mu) &> d_i^\delta(\mu)
\end{aligned}
$$

□

EXAMPLE 3. When externalities are sufficiently low then the efficient network is the one where the types segregate but low type are more central as there are more of them.

Suppose there are seven agents - four of low type (1,2,3,4) and three of high type (5,6,7). There is supermodularity; a degree quota of two, and; a level of network externalities δ. Consider the segregated network $\mu = \{12, 23, 34, 14\} \cup \{56, 67, 75\}$, and the connected network $\tilde{\mu} = \{12, 23, 34, 45, 56, 67, 71\}$. If the inequality below holds with ">" then μ is an efficient network; if "<" then $\tilde{\mu}$ is an efficient network.

$$
3 \cdot Z(\bar{x}, \bar{x}) + 4 \cdot Z(\underline{x}, \underline{x}) \gtreqless 2 \cdot Z(\bar{x}, \bar{x}) + 3 \cdot Z(\underline{x}, \underline{x}) + (2 + 4\delta + 6\delta^2) \cdot Z(\bar{x}, \underline{x}) \tag{31}
$$

When Inequality 31 holds with ">" then μ is furthermore strongly stable when for every pair of agents $i, j \in N; i \neq j$ it holds that there are no transfers between them, i.e. $\tau_{ij} = 0$. In the network μ it holds that $d_i^\delta(\mu) > d_j^\delta(\mu)$ where $i \in \{1, 2, 3, 4\}, j \in \{5, 6, 7\}$. Note that Inequality 31 holds with ">" if $\hat{Z} > 1 + 2\delta + 3\delta^2$ and $\hat{Z} > 1$ due to supermodularity.

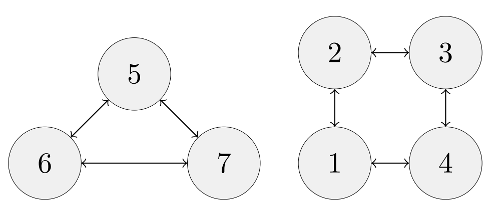

Figure 4: The above networks depict failure of monotonic centrality in the presence of supermodularity (i.e. no absence of modularity) from Example 3.

[21]Both agents have a number of links that do not exceed the degree quota in the original network - thus their degree quota cannot be exceed in the alternative network.

F Sorting in degree

THEOREM 4: If there are no externalities, supermodularity and monotonicity in link value, as well as complete heterogeneity then the set of pairwise stable networks has top sorting in degree.

Proof. As there are no externalities, supermodularity and monotonicity in link value then Theorem 3 and 1 must hold. In addition, as agents i and j are distinct under complete heterogeneity then either $x_i > x_j$ or $x_{i'} > x_i$. As $k_i(\mu) \geq k_j(\mu)$ it must be that $x_i > x_{i'}$ as the converse would violate Theorem 3.

From Theorem 1 it is known that if there are two agents such that $x_i > x_{i'}$ then this entails that for $l = 1, ..., k_j(\mu) : \mathcal{X}(\nu_i(\mu)/\{j\})_l \geq \mathcal{X}(\nu_j(\mu)/\{i\})_l$. This inequality entails there are exactly two possible cases for any index $l = 1, ..., k_j(\mu)$. The first case is that $\mathcal{X}(\nu_i(\mu)/\{j\})_l = \mathcal{X}(\nu_j(\mu)/\{i\})_l$. In this case $\mathcal{K}(\nu_i(\mu)/\{i\})_l = \mathcal{K}(\nu_j(\mu)/\{i\})_l$ as the agent linked to i and j must be the same due to complete heterogeneity. Else in the other case where $\mathcal{X}(\nu_i(\mu)/\{j\})_l > \mathcal{X}(\nu_j(\mu)/\{i\})_l$ then reapplying Theorem entails 3 in which case it follows that, $\mathcal{K}(\nu_i(\mu)/\{i\})_l \geq \mathcal{K}(\nu_j(\mu)/\{i\})_l$. □

G Limits to sorting and monotonic centrality

LEMMA 2. In Example 2 the sufficient conditions for existence of τ such that μ is stable are linear costs ($\tilde{c}$ of forming a link) and δ being sufficiently high along with the two inequalities:

$$2 \cdot Z(\bar{x}, \underline{x}) + Z(\bar{x}, \bar{x}) \geq 4\tilde{c} \quad \text{and} \quad Z(\bar{x}, \underline{x}) > \tilde{c}.$$

Proof. Linear cost entails $\Delta c(1) = c(1)$ and $c(2) = 2c(1)$. Note that the total cost of establishing a link is $2\tilde{c}$. Throughout we assume that $s_2 = s_3$ and $\tau_{23} = 0$ which implies $\tau_{12} = \tau_{13}$. To shorten notation let $s_i = s_i(\mu, \tau)$.

We begin with deriving the required transfers and net utility for agent 1 - the relevant Inequalities are 7, 8, 11. The two inequalities (for $i = 2, 3$) in Inequality 8 can be rewritten using that $\tau_{12} = \tau_{13}$:

$$\begin{aligned} \tau_{12} &\geq \tilde{c} - z(\underline{x}, \bar{x}) \\ \tau_{12} + \tau_{13} &\geq 2\tilde{c} - 2z(\underline{x}, \bar{x}) \\ 2\tilde{c} - 2z(\underline{x}, \bar{x}) + s_1 &\geq 2\tilde{c} - 2z(\underline{x}, \bar{x}) \\ s_1 &\geq 0 \end{aligned} \tag{32}$$

Thus Inequality 8 is irrelevant as it is always satisfied when $\tau_{12} = \tau_{13}$ and Inequality 7 ($s_1 \geq 0$) is satisfied.

By adding the two inequalities in Inequality 11 where $i = 1$ and $j = 2, 3$ together and subtracting Equation 3 the following must hold for the net utility of agent 1:

$$\begin{aligned} s_1 &\geq 2Z(\bar{x}, \underline{x}) - 4\tilde{c} - [\delta Z(\bar{x}, \bar{x}) + 2Z(\bar{x}, \underline{x}) - 4\tilde{c}] \\ s_1 &\geq -\delta Z(\bar{x}, \bar{x}) \end{aligned} \tag{33}$$

We see that Inequality 33 is satisfied when Inequality 7 holds ($s_1 \geq 0$). Note that two inequalities in Inequality 11 where $i = 1$ and $j = 2, 3$ must still hold.

We now turn to derive to check relevant transfer and net utilities for agent 2 and 3. The relevant expressions are Inequalities 7, 11, 10 and 9. For large enough δ (i.e. $\delta \to 1$) then it always holds that Inequality 9 is satisfied; thus it suffices to check Inequalities 7, 11 and 10.

We can use a similar same procedure to see what implications the inequalities have for the net utility of agent $i \in \{2,3\}$. Let $j = \{2,3\} \backslash \{i\}$. By adding the two inequalities from Inequality 11 for the pair $1,i$ and the pair i,j, together and subtracting Equation 3 it must hold that:

$$\begin{aligned} \min\{s_2,s_3\} &\geq Z(\bar{x},\underline{x}) + Z(\bar{x},\bar{x}) - 4\tilde{c} - [\delta Z(\bar{x},\bar{x}) + 2Z(\bar{x},\underline{x}) - 4\tilde{c}] \\ \min\{s_2,s_3\} &\geq (1-\delta)Z(\bar{x},\bar{x}) - Z(\bar{x},\underline{x}) \end{aligned} \tag{34}$$

For sufficiently large δ Inequality 34 becomes irrelevant as in the limit of $\delta \to 1$ its requirement is $\min\{s_2,s_3\} \geq -Z(\bar{x},\underline{x})$; this inequality is always satisfied when $\min\{s_2,s_3\} \geq 0$. Note that it still remains to check all of the Inequality 11's three inequalities.

In addition, it is possible to rewrite Inequality 10 in to Inequality 35. Note that Inequality 35 is a sufficient condition for Inequality 35 for both agent 2 and 3.

$$\begin{aligned} s_2 + s_3 &\geq Z(\bar{x},\bar{x}) + (1+\delta)z(\bar{x},\underline{x}) + \max\{\tau_{21},\tau_{31}\} - 3\tilde{c} \\ s_2 + s_3 &\geq Z(\bar{x},\bar{x}) + (1+\delta)z(\bar{x},\underline{x}) + \max\{s_2,s_3\} - \delta z(\bar{x},\bar{x}) - z(\bar{x},\underline{x}) - 2\tilde{c} \\ \min\{s_2,s_3\} &\geq (2-\delta)z(\bar{x},\bar{x}) + \delta z(\bar{x},\underline{x}) - 2\tilde{c} \end{aligned} \tag{35}$$

In the remainder of this proof we restrict that $s_1 = 0$ and thus $\tau_{21} = \tau_{31} = z(\underline{x},\bar{x}) - \tilde{c}$. This restriction ensures the minimal requirements for the transfer and net utility for agent 1 are met (see Inequalities 7, 8, 11). This restriction allows me to assess when the remaining requirements are satisfied for agents 2 and 3 given that the requirements for agent 1 are minimally satisfied. When $s_1 = 0$ then Inequality 11 is satisfied if: $s_1 = 0$

$$\min\{s_2,s_3\} \geq Z(\bar{x},\underline{x}) - 2\tilde{c} \tag{36}$$

It is possible a sufficient condition for the remaining relevant inequalities being satisfied when $s_2 = s_3$ and $s_1 = 0$. Recall that it is necessary to check Inequalities 7, 11 and 10 for large enough δ. The sufficient condition for the inequalities is captured in Inequality 37 below. The elements in the set from which the maximal element is chosen derived from respectively: Inequality 7; Inequality 11 for the pair 2,3; Inequality 36 which implies Inequality 11 for the pair 1,2 and the pair 1,3; Inequality 35 which implies Inequality 10 is satisfied.

$$s_2 + s_3 \geq \max\{0,\, Z(\bar{x},\bar{x}) - 2\tilde{c},\, 2Z(\bar{x},\underline{x}) - 4\tilde{c},\, 2(2-\delta)z(\bar{x},\bar{x}) + 2\delta z(\bar{x},\underline{x}) - 4\tilde{c}\} \tag{37}$$

Using that $s_1 = 0$ it holds that $s_2 + s_3 = s_1 + s_2 + s_3$. This can be combined by with substituting Equation 3 into Inequality 37. This implies the following four Inequalities - each inequality correspond to an n'th element in the set from which the maximal element is chosen.

$$2Z(\bar{x},\underline{x}) + \delta Z(\bar{x},\bar{x}) - 4\tilde{c} \geq 0 \tag{38}$$

$$\begin{aligned} 2Z(\bar{x},\underline{x}) + \delta Z(\bar{x},\bar{x}) - 4\tilde{c} &\geq 2Z(\bar{x},\underline{x}) - 4\tilde{c} \\ (1-\delta)Z(\bar{x},\bar{x}) &\geq 0. \end{aligned} \tag{39}$$

$$\begin{aligned} 2Z(\bar{x},\underline{x}) + \delta Z(\bar{x},\bar{x}) - 4\tilde{c} &\geq 2(2-\delta)z(\bar{x},\bar{x}) + 2\delta z(\bar{x},\underline{x}) - 4\tilde{c} \\ 2 \cdot (\underline{x},\bar{x}) + 2(1-\delta)(\bar{x},\underline{x}) &\geq 2(1-\delta)Z(\bar{x},\bar{x}) \end{aligned} \tag{40}$$

$$\begin{aligned} 2Z(\bar{x},\underline{x}) + \delta Z(\bar{x},\bar{x}) - 4\tilde{c} &\geq Z(\bar{x},\bar{x}) - 2\tilde{c} \\ 2Z(\bar{x},\underline{x}) &\geq (1-\delta)Z(\bar{x},\bar{x}) + 2\tilde{c} \end{aligned} \tag{41}$$

Of the above we see that Inequality 39 always holds while Inequality 40 holds when δ is large enough. Thus all the above inequalities can be satisfied for large enough δ when $2Z(\bar{x},\underline{x})+\delta Z(\bar{x},\bar{x}) \geq 4\tilde{c}$ and $Z(\bar{x},\underline{x}) > \tilde{c}$. □

CONDITION 1. For $s_1, s_2, ..., s_n$ it is required that the following equation holds:

$$\Sigma_{i\in N} s_i \quad = \quad U(\mu) = \Sigma_{i\in\{2,..,n\}} Z_{i1} + \Sigma_{i,j\in\{2,..,n\},i>j} \delta Z_{ij} - (n-1)\cdot c(1) - c(n-1), \tag{42}$$

along with the following three set of inequalities,

$$\forall i \in N: \quad s_i \geq 0, \tag{43}$$
$$\forall i \neq 1: \quad \tau_{1i} \geq \Delta c(n-2) - z_{1i}, \tag{44}$$
$$\forall i,j \in N, i\neq j: \quad s_i + s_j \geq Z_{ij} - 2\cdot c(1). \tag{45}$$

and finally the following two set of inequalities must hold for any two $i,j \in (N\backslash\{1\}), i\neq j$:

$$s_i + s_j \geq Z_{ij} + z_{i1} + \delta z_{j1} + \delta\Sigma_{l\notin\{1,i,j\}}[z_{il} + \delta z_{jl}] - c(1) - c(2) + \tau_{i1}, \tag{46}$$
$$2\cdot\Delta c(2)x \geq (1-\delta)\cdot Z_{ij}. \tag{47}$$

In what follows we demonstrate two auxiliary results: Lemma 3 and Lemma 4. These two results are used in demonstrating Proposition 5 and thus Corollary 1. The first auxiliary lemma rely on Condition 1 above.

LEMMA 3. Suppose there are at least three agents ($n \geq 3$), weak supermodularity and monotonicity, a cost function and an interval $[\tilde{\delta},\hat{\delta}]$ where Condition 1 is satisfied, then for every $\delta \in [\tilde{\delta},\hat{\delta}]$ the network μ where a least talented agent is the center of a n-1-star (i.e. $\mu = \{12, 13, ..., 1n\}$) is pairwise stable.

Proof. Let $\mu = \{12, 13, ..., 1n\}$ which is a network where the least talented is a center of a (n-1)-star network. The aim is to show that a (n-1)-star network with the least talented as center may be pairwise stable in the following setting: there is weak supermodularity and monotonicity, there are at least three agents ($n \geq 3$) along with Condition 1.

First we check for deviations where an agent deletes all its links in μ. This move cannot be profitable when Inequality 43 is satisfied for all individuals. This implies for any agent but agent 1 it is not profitable to delete their single link in μ. Furthermore for agent 1 deleting any link is not profitable due to due to Inequality 44 being satisfied. But this also implies that deleting any number of links for agent 1 is not beneficial as they are all beneficial at the highest marginal cost.

In addition any two agents can at least secure the value of the link that they can form independently of other links which is ensured by Inequality 45.

Finally agents apart from agent 1 can form a link between themselves while keeping one or both of their links to agent 1. The network μ is robust to deviations where one agent deletes a link with 1 but forms a new to another agent that keeps its link with agent 1 if Inequality 46 is satisfied. The network μ is robust to deviations where both agents keep their current link with agent 1 and form a new link together if Inequality 47 is satisfied. □

LEMMA 4. For Lemma 3 the requirement of Condition 1 being satisfied may be replaced with linear costs of linking and sufficiently many agents. For large number of agents it holds that:

$$s_i(\mu,\tau) \geq (1-\delta^2)Z_{in} - (\delta-\delta^2)z_{ni} + \delta z_{i1} + \Sigma_{l\notin\{1,i\}}\delta^2 z_{il} - 2\tilde{c}, \quad i\neq 1$$
$$s_1(\mu,\tau) \in [0,\ (1-\delta)U(\mu) + \Sigma_{i\in\{2,..,n\}}[(\delta-\delta^2)z_{ni} - (1-\delta^2)Z_{in} + \delta z_{1i}]]$$

where $\mu = \{12, .., 1n\}$ and τ is consistent with the inequalities above.

Proof. Let the setting be as specified by Lemma 3 with the following cost technology. The costs are linear such that each link has some cost, i.e., for any degree $k \in \mathbb{N}_0$: $\Delta c(k) = \tilde{c}$ which is cost of establishing a link - thus the total costs are $c(k) = k \cdot \tilde{c}$. Note that the total cost of establishing a link then is $2\tilde{c}$ as two agents are required.

Due to weak supermodularity and monotonicity it holds that there is an lower and upper bound on link value for any $i, j : Z(\bar{x}, \bar{x}) \geq Z_{ij} \geq Z(\underline{x}, \underline{x})$. We will use these two properties along with the condition of many agents to show there exist some threshold where the (n-1)-star network with 1 is pairwise stable.

We can use a procedure similar to the one used in Example 2 to find the net utility of agent $i \in N$. By adding the n-1 inequalities from Inequality 45 for i with all other agents (i.e. in $N\backslash\{i\}$) together and subtracting Equation 42 the expression in Inequality 48 must hold.[22]

$$\min_{i\in\{2,..,n\}} s_i \quad \geq \quad Z(\bar{x}, \underline{x}) + (n-2)Z(\bar{x}, \bar{x}) - [(n-1) + \delta(n-2)^2] \cdot Z(\underline{x}, \underline{x})] \tag{48}$$

For $\delta > 0$ it holds that for sufficiently large n then Inequality 48 becomes irrelevant as for large n the right hand side take values below zero for all $i \in N$.

In addition, for any two agents $i, j \neq 1$ it is possible to rewrite Inequality 46 in to the following:

$$\begin{aligned} s_i + s_j &\geq Z_{ij} + z_{j1} + \delta z_{i1} + \Sigma_{\iota\notin\{1,i,j\}}[\delta z_{j\iota} + \delta^2 z_{i\iota}] - 3\tilde{c} + \tau_{j1} \\ s_i + s_j &\geq Z_{ij} + \delta z_{i1} + \Sigma_{\iota\notin\{1,i,j\}}\delta^2 z_{i\iota} - 2\tilde{c} - \delta z_{ji} + s_j \\ s_i &\geq z_{ij} + (1-\delta)z_{ji} + \delta z_{i1} + \Sigma_{\iota\notin\{1,i,j\}}\delta^2 z_{i\iota} - 2\tilde{c} \end{aligned} \tag{49}$$

The maximum of Inequality 49 for agent i is:

$$\begin{aligned} s_i &\geq \max_{j\notin\{1,i\}} \left[z_{ij} + (1-\delta)z_{ji} + \delta z_{i1} + \Sigma_{\iota\notin\{1,i,j\}}\delta^2 z_{i\iota}\right] - 2\tilde{c} \\ s_i &\geq \max_{j\notin\{1,i\}} \left[(1-\delta^2)z_{ij} + (1-\delta)z_{ji}\right] + \delta z_{i1} + \Sigma_{\iota\notin\{1,i\}}\delta^2 z_{i\iota} - 2\tilde{c} \\ s_i &\geq \max_{j\notin\{1,i\}} \left[(1-\delta^2)Z_{ij} - (\delta-\delta^2)z_{ji}\right] + \delta z_{i1} + \Sigma_{\iota\notin\{1,i\}}\delta^2 z_{i\iota} - 2\tilde{c} \end{aligned} \tag{50}$$

A sufficient condition for Inequality 50 is that (as $x_n = \bar{x}$ and there is weak supermodularity and monotonicity):

$$s_i \quad \geq \quad (1-\delta^2)Z_{in} - (\delta-\delta^2)z_{ni} + \delta z_{i1} + \Sigma_{l\notin\{1,i\}}\delta^2 z_{il} - 2\tilde{c} \tag{51}$$

For any $\delta > 0$ it holds that as $n \to \infty$ then $\sum_{l\notin\{1,i\}} \delta^2 z_{il} \to \infty$ for any $i \in \{2, .., n\}$. Thus, it follows that as $n \to \infty$ the only relevant condition among the above is Inequality 51 as the sum term will dominate all other. This implies that for a large number of agents the requirement on net for agents excluding agent 1 are:

$$\Sigma_{i\in\{2,..,n\}} s_i \quad \geq \quad \Sigma_{i\in\{2,..,n\}}[(1-\delta^2)Z_{in} - (\delta-\delta^2)z_{ni} + \delta z_{i1}] + \Sigma_{i,j\in\{2,..,n\},i>j}\delta^2 Z_{ij} - (2n-2)\tilde{c} \tag{52}$$

[22] Inequality 48 uses the upper bound on the value for the links for agent i with others (from Inequality 45) and the lower bound on the total value of the network (from Equation 42). Note that costs in the n-1 inequalities are equal to those from Equation 42 - so these have zero net effect.

We can check when Inequality 52 is possible to with non-waste-fullness from Equation 42:

$$\begin{aligned}
s_1 &= \Sigma_{i\in\{1,..,n\}} s_i - \Sigma_{i\in\{2,..,n\}} s_i \\
&\leq \Sigma_{i\in\{2,..,n\}} Z_{i1} + \Sigma_{i,j\in\{2,..,n\},i>j} \delta Z_{ij} - (n-1)\cdot c(1) - c(n-1) \\
&\quad -[\Sigma_{i\in\{2,..,n\}}[(1-\delta^2)Z_{in} - (\delta-\delta^2)z_{ni} + \delta z_{i1}] + \Sigma_{i,j\in\{2,..,n\},i>j}\delta^2 Z_{ij} - (2n-2)\tilde{c}] \\
&= \Sigma_{i\in\{2,..,n\}}[Z_{i1} + (\delta-\delta^2)z_{ni} - (1-\delta^2)Z_{in} - \delta z_{i1}] + \Sigma_{i,j\in\{2,..,n\},i>j}(\delta-\delta^2)Z_{ij} \\
&= \Sigma_{i\in\{2,..,n\}}[(1-\delta)Z_{i1} + (\delta-\delta^2)z_{ni} - (1-\delta^2)Z_{in} + \delta z_{1i}] + \Sigma_{i,j\in\{2,..,n\},i>j}(\delta-\delta^2)Z_{ij} \\
&= (1-\delta)U(\mu) + \Sigma_{i\in\{2,..,n\}}[(\delta-\delta^2)z_{ni} - (1-\delta^2)Z_{in} + \delta z_{1i}]
\end{aligned}$$

The n-1 inequalities represented by Inequality 44 for agents $2,..,n$ will hold if $\tau_{1i} = \tilde{c} - z(\underline{x}, x_i)$ and their aggregate inequality holds (as setting $\tau_{1i} = \tilde{c} - z(\underline{x}, x_i)$ will maximize the net utility for agent i subject to Inequality 44 holding):

$$\Sigma_{i\neq 1}\tau_{1i} = (n-1)\tilde{c} - \Sigma_{i\neq 1} z(\underline{x}, x_i) \quad (53)$$

$$\Updownarrow$$

$$s_1 = 0 \quad (54)$$

By adding the n-1 inequalities in Inequality 45 for 1 with $2, 3..., n$ together and subtracting Equation 3 the following must hold for the net utility of agent 1:

$$s_1 \geq -\Sigma_{i,j\in\{2,..,n\},i>j}\delta Z_{ij} \quad (55)$$

We see the condition in Inequality 43 is always stronger than that in Inequality 55 - thus Inequality 55 is irrelevant and the only condition for agent 1 is that $s_1 \geq 0$.

□

PROPOSITION 5: For any number of agents if there are externalities, monotonicity, supermodularity and a cost function then the network where the least type is center of a star is inefficient but pairwise stable under certain conditions.

Proof. The conditions can be found in Lemma 3 and Lemma 4. □