# The Government Expenditure Multiplier in an Endogenous Switching Model*

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Preliminary and incomplete
August 2011

**Abstract**

This model analyses the output multiplier of government expenditure when the nominal interest rate is at zero and imposes the assumption that the level of government expenditure directly affects the probability that the economy will exit the liquidity trap. This is done in a New Keynesian sticky price setup. A multiplier well above one is obtained when the monetary policy is connstrained by the zero lower bound and is further increased when the fiscal stimulus causes a greater probability of exiting the liquidity trap.

*Keywords:* DSGE Modelling, Endogenous markov switching, Fiscal Policy, Liquidity trap, Monetary Policy, Nominal interest rate zero lower bound .

*JEL classification*: .

In the wake of the recent financial crisis a heated debate has erupted over the efficiency of fiscal policy as a means to stimulate an economy that finds itself in a liquidity trap [1]. As many of the great economies found their traditional monetary policy constrained at the zero lower bound and thus incapable of stimulating the economy through the traditional interest rate channel, the argument often used is that fiscal expansion causes the real rate to fall since the nominal interest rate cannot be increased, as it would under an unconstrained Taylor rule thereby crowding in private consumption. The ongoing debate has treated the probability of exiting the liquidity trap (and thus also the expected duration of the trap) as an exogenous Markov process. This paper argues that through holding a hand under the financial sector, perhaps even alleviating some of the credit frictions, the government can potentially reduce the duration of the trap, so the Markov process is assumed to be a function of the government's consumption.

This paper takes the offset in the Woodford (2010) paper and

*The author wishes to thank Henrik Jensen for competent and inspiring guidance. All remaining errors are my own. The views expressed are those of the author, and do not necessarily correspond to those of Danmarks Nationalbank.

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[1] In New Keynesian economics the term liquidity trap refers to a situation where monetary policy cannot be used to stimulate the economy. In this paper we will use the term liquidity trap and being at the zero lower bound interchangeably

In August 2011 the US Debt Crisis took headlines worldwide, as the parties in Congress struggled to reach an agreement on how to deal with the debt ceiling becoming binding. The large debt was amongst other things a result of increased government expenditures as a means of stimulating the economy that was in a slump. Only shortly after an agreement was found, did the financial markets start pricing a considerable credit premium and the rating agency StandardandPoors shortly thereafter downgraded American government debt from a AAA rating (the highest possible) to AA+. Negative sentiments amongst investors about the time horizon for exiting the crisis caused the markets to bleed, and shortly hereafter the Federal Open Market Committee released a statement saying that they expected their interest rate to remain at 0-0.25 pct. p.a. for the next two years, which is much more severe than what the "for an extended period" phrasing up to this point had indicated. [2] This case as well as many other examples from the ongoing European debt crisis show that the fiscal policy can indeed affect the duration of the crisis in a very profound way.

I begin by reviewing the literature on the fiscal multiplier at the zero lower bound in section 1. In Section 2 I set up a simple New Keynesian sticky price model, in this benchmark model the multiplier is greater than one when the monetary policy is at the zero lower bound. This is due to the effect that the fiscal expansion causes the real rate to fall when the interest rate cannot be increased, as it would under an unconstrained Taylor rule. Section 3 considers a simple two state Markov model, where credit frictions cause the credit spread to increase and thus the economy finds itself in a liquidity trap, with a constant markov probability of exiting this trap (when spreads return to normal). Section 4 introduces the link between government expenditure and the markov probability of exciting the liquidity trap, and shows that this can have a large effect on the multiplier. Section 5 briefly discusses the critical assumptions, including the consequence of extending the model with distortionary taxes. Section 6 summarizes the paper's conclusions.

# 1 Literature on the Government Spending Multiplier and on Endogenous Switching

**brief sketch**

- Eggertsson (2010) finds a multiplier after a marginal increase in government spending of 2.3 in a liquidity trap compared to 0.3 when the short term interest rate is positive. The intuition is the same as stated above. Eggertsson (2010) also extensively looks into the effects of fiscal multipliers when using tax cuts and concludes that the effect of using fiscal stimulus is much larger when aiming at the demand side rather than the supply side.
- Davig and Leeper (2008) investigate the effect of an endogenous monetary policy. They consider a policy rule, where the parameters switch once inflation and output reach certain levels and thus have the zero lower bound policy rule nested as a threshold switching rule within their larger family of policy rules. As Davig and Leeper (2008) point out themselves that there is no sharp conceptual distinction between an endogenous regime change in monetary policy and a nonlinear monetary policy rule, so this paper should not be seen as discussing the same concept, as in the present paper.

[2] "The Committee currently anticipates that economic conditions–including low rates of resource utilization and a subdued outlook for inflation over the medium run–are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013. " FOMC statement August 2009.

- Christiano et al. (2009) estimate the size of the government expenditure multiplier at the zero lower bound and find that this is an impressive 3.7.
- Mertens and Ravn (2010) take much of the approach used in Christiano et al. (2009) but do not linearize the model around the equilibrium. Further, their liquidity trap is caused by a non-fundamental shift ti the sunspot equilibrium in the liquidity trap (this can exist a second equilibrium in the economy as long as there is a positive probability of switching to the intended zero-inflation steady state). By arguing that the current crisis is caused by a nonfundamental shock to the sunspot variable, they find a fiscal spending multiplier of 0.23 when the economy is in a liquidity trap, which is significantly lower than their estimated multiplier of 0.55 in the steady state.

# 2 Model

We initially consider a New Keynesian log-linearized model,which can be solved analytically when we do not have endogenous markov switching. Since most recent articles discussing the Government expenditure multiplier (except Mertens and Ravn (2010) and Benhabib et al. (2002)[3]) use this log-linear approach, we will do so in order to look isolated at the effect of the endogenous switching in this setup, rather than using the more extensive but less used second order setup.

## 2.1 Households

We consider an economy made up of a large number of identical, infinitely lived households each of which seeks to maximize

$$U = \sum_{t=0}^{\infty} \beta^t \left[ u\left(C_t\right) - v\left(H_t\right) \right] \tag{1}$$

where $C_t$ is the is consumption in period $t$ of the economy's single produced final good, $H_t$ is hours of labour supplied by the household in period $t$, the instantaneous utility functions satisfy $u' > 0, u'' < 0, v' > 0, v'' < 0$, and the discount factor satisfies $0 < \beta < 1$. The household is subject to a standard transversality condition. The additive separability of consumption and supplied labor in the utility function [4]

The optimal consumption path for the household satisfies the Euler condition

$$\frac{u'\left(C_t\right)}{\beta u'\left(C_{t+1}\right)} = e^{r_t} \tag{2}$$

where $r_t$ is the continuously compounded real rate of return between $t$ and $t+1$. It directly follows from (2) that in steady state, $r_t = \overline{r} \equiv -\log \beta > 0$ in each period. This real interest rate is the only one at which the economy converges asymptotically to a steady state.

[3] Benhabib et al. (2002) are different than the other papers in three ways. First, instead of analyzing liquidity traps with stochastic duration they asssume permanent liquidity traps in a perfect foresight context. Second, they look at an endowment economy as opposed to our production economy. Finally, they assume perfect price flexibility.

[4] Christiano et al. use estimate their model with and without additively seperable utility and find that "across a wide set of parameter values, dY=dG is always less than one with this preference specification", whereas they get a multiplier larger than one when they assume complementarity of consumption and leisure in preferences.

The optimal level of labor supply for the household satisfies the first order condition

$$\frac{v'(H_t)}{u'(C_t)} = \frac{W_t}{P_t} \tag{3}$$

where $W_t$ is the is nominal wage in period $t$ and $P_t$ is the price of the final good.

## 2.2 Firms

**Final Goods Producers** In the economy there is a homogeneous final good, which is sold in a competitive market and produced from a continuum of differentiated intermediate goods through a constant elasticity of substitution (CES) technology

$$Y_t = \left( \int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} \, di \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{4}$$

where $Y_t(i)$ is the quantity used of intermediate good $i$, and $\epsilon > 1$ is the elasticity of substitution between the intermediate goods. The price of the final good is fully flexible, and since we assume that there are no adjustment costs, the problem of maximizing the present value of future profits reduces to simply maximizing profits in each period,

$$\max_{Y_t(i)} P_t \left( \int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} \, di \right)^{\frac{\epsilon}{\epsilon-1}} - \int_0^1 p_t(i) \, Y_t(i) \, di \tag{5}$$

where $p_t(i)$ is the price of the intermediate good $i$. This results in the inverse demand function

$$p_t(i) = P_t \left( \frac{Y_t}{Y_t(i)} \right)^{\frac{1}{\epsilon}} \tag{6}$$

where $\epsilon$ is the elasticity of demand.

The final good is either consumed by the households or by the government, which gives us the equilibrium condition

$$Y_t = C_t + G_t. \tag{7}$$

### 2.2.1 Intermediate firms

The intermediate goods are sold in a market with monopolistic competition and each of the goods is produced using a constant returns to scale technology

$$Y_t(i) = k_t(i) f\left(h_t(i) / k_t(i)\right) \tag{8}$$

where $k_t(i)$ is the is amount of capital used by firm $i$, $h_t(i)$ are hours of labor hired by that firm, and $f(\cdot)$ is a production function satisfying $f' > 0, f'' < 0$. This is the familiar Dixit-Stiglitz model of monopolistic competition (Dixit and Stiglitz (1977)).

We assume that the total amount of capital used by the intermediate firms is an exogenous fixed amount, which we normalize to one. The capital is allocated between the firms in a competitive rental market. We then have that each firm will have the same marginal cost of production in a given period, which will be a homogenous degree 1 function of the two competitive prices on labour and capital. This homogeneity means that cost-minimization will imply the same kapital-labor ratio for each intermediate firm, regardless of its scale of production.

Given our normalization of capital, the equilibrium value of this kapital-labor ratio must equal the aggregate labor supply $H_t$. Assuming a perfectively competitive labor market and flexible wages, the labor market clearing condition becomes

$$\frac{MC_t}{P_t} = \frac{v'(H_t)}{u'(C_t)} \tag{9}$$

If prices were fully flexible, profit maximization for a firm facing a downward sloping demand curve with elasticity of demand $\epsilon$ will set the price as a constant markup over the marginal cost so that

$$MC_t = W_t / f'(H_t) \tag{10}$$

where $MC_t$ is the nominal marginal cost and $\mu \equiv \frac{\epsilon}{1-\varepsilon} > 1$.Given that the price-markup is constant, this wedge will not have an effect on the Government expenditure multiplier. If however we assume that either the wage setting or price setting is sticky, there will be an endogenous variation in the markup (or labor efficiency wedge in the case of sticky wages) thereby creating demand side effects, which should also be considered when determining the size of the government expenditure multiplier.

We assume instead that the prices are Calvo style rigid, and each intermediate firm can in each period with a probability $1-\theta$ freely reset its price but with the probability $\theta$ has to maintain the price at what is was the previous period, $\geq$. The probabilities are assumed exogenous, i.e. independent of the time of the last price change of the firm or the current price. This is indeed a very simplifying assumption, but it greatly reduces the state space required to analyze the dynamic outcome of the economy.

The profit maximization problem of the of the intermediate firm is

$$\max_{\{p_t(i), Y_t(i), N_t(i)\}} E_0 \sum_{t=0}^{\infty} \beta^t \lambda_t \left(p_t(i) Y_t(i) - (1-v) W_t h_t(i)\right) \tag{11}$$

subject to (6), (8), and the price rigidity condition $p_t(i) = p_{t-1}(i)$ with probability $\theta$. Since the firms are owned by the households, the instantaneous shadow

price on consumption, $\lambda_t$, is used together with the rate of time preference $\beta$ to discount profits. To offset inefficiencies in the labour market that would arise from the imperfect competition between intermediate firms, these receive a subsidy $v = \frac{1}{\epsilon}$ per labor unit, thus removing the price markup. We note that any firm that has the opportunity to reset its price will face the same demand-function and technology, thus the optimal reset price will be firm independent.

The optimality condition for the reset price is then

$$p_t^* = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \lambda_{t+j} Y_{t+j}(i) MC_{t+j} P_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \lambda_{t+j} Y_{t+j}(i)} \tag{12}$$

From the technology defined in (4) combined with the demand function (6) we get the aggregate price level related to the prices in the intermediate market by

$$P_t = \left( \int_0^1 p_t(i)^{1-\epsilon}\, di \right)^{\frac{1}{1-\epsilon}}. \tag{13}$$

which is the standard Dixit-Stiglitz index for the aggregate price level. Using this together with the production technology in the intermediate market 8, we have the following relationship between final output and labor input.

$$Y_t = D_t^{-1} N_t \tag{14}$$

where $D_t = \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{-\epsilon} di \geq 1$ is a measure of the price dispersion in the economy. Since price dispersion causes final goods producers to change their relative weight of intermediate goods in their production function, despite the fact that the intermediate firms all have a linear production technology and will have equal weights in a flexible price model . This inefficient allocation of labour is the reason why price dispersion reduces the total level of output for a given level of labour.

Finally, by using the fact that each intermediate price is Bernoulli distributed with outcome $P_{t-1}(i)$ and $P_t^*(i)$ with probability $\theta$ and $1-\theta$, we can use equation (13) to write

$$(1+\pi_t)^{1-\epsilon} = \theta + (1-\theta) \left( \frac{p_t^*}{P_{t-1}} \right)^{1-\epsilon}. \tag{15}$$

## 2.3 Monetary and Fiscal Policy

Letting $\widehat{Y}_t = \log \frac{Y_t}{\overline{Y}}$ and $\widehat{G}_t = \log \frac{G_t}{\overline{Y}}$ be the relative deviations of output and government expenditure from the deterministic steady state value of output $\overline{Y}$. This ensures comparability in units between the two relative deviations and furthermore that $\widehat{G}_t$ is defined even when the steady state value of government expenditure is zero.

The monetary policy is assumed to set the nominal interest rate according to a Taylor rule subject to a zero lower bound[5]

$$i_t = \max \left\{ 0, \overline{r} + \phi_\pi \pi_t + \phi_y \left( \widehat{Y}_t - \Gamma \widehat{G}_t \right) \right\} \tag{16}$$

where $i_t$ is a nominal short term risk free interest rate under control of the central bank, $\overline{r}$ is the nominal rate that is consistent with a zero inflation steady state, and the policy response coefficients satisfy $\phi_\pi > 1, \phi_y > 0$ so that the Taylor principle holds (Taylor (1993)). $\Gamma$ is the flexible-price multiplier in a model with perfect competition[6], so that $\widehat{Y}_t - \Gamma \widehat{G}_t$ is the output gap with regards to the flexible price equilibrium (see next section). This is consistent with the notion above that the real interest rate must equal $\overline{r}$ in the zero-inflation steady state.

[5] Woodford (2011) switches between using the term $\phi_y \left( \widehat{Y}_t - \Gamma \widehat{G}_t \right)$ and $\phi_y \widehat{Y}_t$ in the Taylor rule. Given the forwardlooking nature of our model, we have that the zero-inflation steady state and the zero-interest steady state will be invariant to this change, only the critical value for $\widehat{G}_{crit}$ (for being at the zero bound) will be affected by the choice between these two policy specifications.

[6] $\Gamma = \frac{\eta_u}{\eta_u + \eta_v} < 1$ where $\eta_u \equiv -\overline{Y} \frac{u'}{u''} > 0$ $\eta_v \equiv \overline{Y} \frac{\widetilde{v}'}{\widetilde{v}''} > 0$ are the elasticities of the two utility functions with regard to an increase in Y. This follows from the labor market clearing condition $u(Y_t - G_t) = \widetilde{v}(Y_t)$ under perfect compeition.

We assume that the government finances its consumption by lump sum taxes, so that a change in the path of government consumption $\{G_t\}$ is always accompanied by an appropriate increase in collected taxes in order to ensure that the government satisfies a transversality condition. We therefore disregard the timing of these lump sum taxes given that Ricardian equivalence holds in this setup. In the deterministic model we assume that the government expenditure will exponentially decrease towards zero

$$\widehat{G}_t = \widehat{G}_0 \rho^t \tag{17}$$

where $0 \leq \rho < 1$, which has the implication that the current level of government expenditure $\widehat{G}_t$ determines the entire future path for government expenditure. Later we introduce a credit shock in the model and then we will respecify this fiscal policy rule, but the essence of the rule will remain the same [7].

## 2.4 Linearization of the model

We follow the tradition of log-linearizing around the zero inflation deterministic steady state. We use the result in Galí (2008) that in a neighborhood of the zero-inflation steady state

$\widehat{D}_t$ is equal to zero up to a first order approximation.

The log-linearized versions of the optimality conditions (12) and (15) are

$$\log p_t^* = \log \mu + (1 - \beta\theta) \sum\nolimits_{j=0}^{\infty} (\beta\theta)^j E_t \left[\log MC_{t+j}\right] \tag{18}$$

$$\log P_t = \theta \log P_{t-1} + (1 - \theta) \log p_t^* \tag{19}$$

where $\mu \equiv \frac{\epsilon}{1-\varepsilon} > 1$.Combining these two equations yield

$$\log \frac{p_t^*}{P_t} = (1 - \beta\theta) \sum\nolimits_{j=0}^{\infty} \beta^j E_t \left[\log \mu + \log MC_{t+j} - \log P_{t+j}\right] \tag{20}$$

The log-linear approximation of the labor market clearing condition is

$$\log MC_t - \log P_t = -\log \mu + \eta_v \widehat{Y}_t + \eta_u \left(\widehat{Y}_t - \widehat{G}_t\right) \tag{21}$$

where . By substituting this into (20) we

$$\log \frac{p_t^*}{P_t} = (1 - \beta\theta) \sum\nolimits_{j=0}^{\infty} \beta^j E_t \left[(\eta_v + \eta_u) \left(\widehat{Y}_{t+j} - \Gamma \widehat{G}_{t+j}\right)\right] \tag{22}$$

where $\Gamma = \frac{\eta_u}{\eta_v + \eta_u} < 1$ is the Government expenditure multiplier in the flex-price version of our model as $\eta_u \equiv -\overline{Y} \frac{u'}{u''} > 0$ $\eta_v \equiv \overline{Y} \frac{\widetilde{v}'}{\widetilde{v}''} > 0$ are the elasticities of the two utility functions with regard to an increase in $Y_t$ [8]. Substituting this into our price level equation (19) gives the following dynamics for the inflation rate

[7] Later we introduce a credit spread that is a Markov chain and and let the Government expenditure $\widehat{G}_t$ stay at a constant level during the existence of the high spread but specify that $\widehat{G}_t$ returns to zero as soon as the disturbance i gone. In this setup the expected level of $\widehat{G}_t$ will be exponentially decaying just as in the present deterministic model, thus the two fiscal policy rules are very similar.

[8] This follows from the labor market clearing condition $u(Y_t - G_t) = \widetilde{v}(Y_t)$ under perfect compeition.

$$\pi_t \equiv \log \frac{P_t}{P_{t-1}} = \frac{1-\theta}{\theta} \log \frac{p_t^*}{P_t} \tag{23}$$

which we can combine with (22) to get that

$$\pi_t = \kappa \sum\nolimits_{j=0}^{\infty} \beta^j E_t \left[ \widehat{Y}_{t+j} - \Gamma \widehat{G}_{t+j} \right] \tag{24}$$

where $\kappa = (1-\theta)(1-\beta\theta)(\eta_u + \eta_v)/\theta > 0$.

Linearizing the monetary Taylor rule in (16) and the Euler rule in (2) yields.

$$i_t = \max \left\{ 0, \overline{r} + \phi_\pi \pi_t + \phi_y \left( \widehat{Y}_t - \Gamma \widehat{G}_t \right) \right\} \tag{25}$$

$$\widehat{Y}_t - \widehat{G}_t = E_t \left( \widehat{Y}_{t+1} - \widehat{G}_{t+1} \right) - \sigma \left( i_t - E_t \pi_{t+1} - \overline{r} \right) \tag{26}$$

where $\sigma \equiv \eta_u^{-1} > 0$ is the intertemporal elasticity of the substitution of private expenditure.

The equilibrium will be the solution to the system of linear equations in (24), (25), and (26). In the appendix are the derivations of the linear equilibrium. Part of the conjectured solution is the equation

$$\widehat{Y}_t = \gamma_y \widehat{G}_t \tag{27}$$

which has the solution

$$\gamma_y = \frac{1-\rho+\psi\Gamma}{1-\rho+\psi} \tag{28}$$

where $\psi = \sigma \left[ \phi_y + \frac{\kappa}{1-\beta\rho} (\phi_\pi - \rho) \right] > 0$ [9]. If there monetary policy did not respond to the output gap (i.e. $\phi_y = 0$) equation (**??**) shows that the government expenditure multiplier $\gamma_y$is necessarily larger than under perfectly flexible prices, as $\Gamma < \gamma_y < 1$. The fact that the monetary policy does respond to the output gap reduces the size of the multiplier, which can can be smaller than in the neoclassical flex-price model if the monetary response to the output is very strong ($\phi_y$ is high). In this case the increase in government expenditure causes such a large real interest rate increase, that the total output in the economy will increase less than under the flex-price case.

In the deterministic model there will be a unique steady state where $\widehat{Y}_t = \widehat{G}_t = 0$ and $\pi_t = 0$ for all $t$.

# 3 A Simple Two State Markov Model

We assume that there is a wedge $\Delta_t$ between the central banks policy rate $i_t$ and the interest rate that households are face when choosing their optimal (expected) consumption path. Such a spread could arise because due to many different inefficiencies in financial intermediation [10] and may vary over time. This setup follows Eggertsson (2010). In times where the inefficiency is very high, , if the zero lower bound becomes binding.

We then have that the log-linear version of the household's Euler equation in (26)

[9] Had we assumed taht the Taylor rule had a constant intercept, the government expenditure multiplier would be $\gamma_y = \frac{1-\rho+(\psi-\sigma\phi_y)\Gamma}{1-\rho+\psi}$.

[10] blabla Curdia and Woodford 2009 blabla.

$$\widehat{Y}_t - \widehat{G}_t = E_t\left(\widehat{Y}_{t+1} - \widehat{G}_{t+1}\right) - \sigma\left(i_t - E_t\pi_{t+1} - r_t^{net}\right) \tag{29}$$

where we define $r_t^{net} = \overline{r} - \Delta_t$,which is the real policy rate that ensures a constant path for private expenditure. From this it becomes apparent that a period of high financial inefficiency may cause $r^{net}$ to be negative, so that the zero bound on the nominal policy rate $i_t$ becomes binding and the central bank finds itself unable to affect private expenditure decisions in the degree desired. In other words it becomes impossible for the central bank to obtain a steady state with zero inflation given that the government expenditure stays at the natural steady state level $\overline{G}$.

We now consider a situation where there has been such a significant substantial financial disruption causing the interest rate spread $\Delta_t$ to have the value $\Delta^L$ whereby $r_t^{net}$ falls to the value $r_L < 0$. We then assume that the financial disturbance is an absorbing Markov

process, so that with probability $0 < \mu < 1$ the financial disruption continues to have the same level in the following period and with probability $1 - \mu$ the credit spread returns to their normal level (here normalized to zero) so that $r_t^{net} = \overline{r}$and remain at this level in all subsequent periods.

As we are interested in the effect of fiscal stimulus when monetary policy is constrained at the zero lower bound, we assume that the fiscal policy will set the level of government expenditure at a high level $G_L$ for the duration of the financial disruption, but returns to the level $\overline{G}$ as soon as the economy exits the liquidity trap. Let $T$ be the the random date at which the credit spread returns to the zero and stays there. As the uncertainty in the economy originates solely from the Markov process for the credit spread, we know that from the date $T$ and forward there is no remaining uncertainty, and hence for any date $t \geq T$ the economy is in a locally determinate unique steady state [11], the zero-inflation steady state where $i_t = \overline{r} > 0$ and $\pi_t = \widehat{Y}_t = 0$.

Since we know this equilibrium for date $T$ and forward, we can solve for the equilibrium that persists while the interest rate spread is high. There exists a forward looking solution to the system of equations (24), (29), and (25) has a bounded solution if and only if the model parameters satisfy [12]

$$\sigma\mu\kappa < (1-\mu)(1-\beta\mu) \tag{30}$$

where $\kappa = (1-\theta)(1-\beta\theta)(\eta_u + \eta_v)/\theta > 0$. This inequality holds if $0 \leq \mu < \overline{\mu} < 1$ for the upper bar $\overline{\mu}$, which depends on the model parameters $\beta$, $\sigma$, and $\kappa$.In other words our model has a bounded solution if the expected duration of the liquidity trap is not too large. We proceed this paper by considering the case where the condition (30) holds, that is where the persistence of the credit friction is not too long.

Given the level of government expenditure $\widehat{G}_L$and the fixed Markov probability $\mu$, the agents are faced with the same probability distribution of future evolutions fundamentals $\left\{r_t^{net}, \widehat{G}_t\right\}$ for all $t < T$. This implies that there is a unique bounded solution, and we find this by using the Markov probabilities, the constant values $\widehat{Y}_L$, $\widehat{G}_L$, and $\pi_L$ and the fact that after the credit spread falls back to zero we have $\widehat{Y}_t = \widehat{G}_t = 0$ . Then (24) and (29)

[11] This is a result of the forward looking nature of equations (24), (29), and (25).
[12] This condition will become obvious in equation (34).

become[13]

$$\pi_L = \frac{\kappa}{1-\beta\mu}\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] \tag{31}$$

$$(1-\mu)\left(\widehat{Y}_L - \widehat{G}_L\right) = -\sigma\left(i_L - \mu\pi_L - r_L\right). \tag{32}$$

These two equations imply that

$$\widehat{Y}_L = \vartheta_r\left(r_L - i_L\right) + \vartheta_G\widehat{G}_L \tag{33}$$

where

$$\vartheta_r = \sigma\frac{(1-\beta\mu)}{(1-\mu)(1-\beta\mu) - \sigma\mu\kappa} > 0 \tag{34}$$

$$\vartheta_G = \frac{(1-\mu)(1-\beta\mu) - \sigma\mu\kappa\Gamma}{(1-\mu)(1-\beta\mu) - \sigma\mu\kappa} > 1. \tag{35}$$

Here we see that the inequalities hold if and only if the requirement for boundedness in (30) holds. By combining equations (33), (25), and (31) and solving for $i_L$(see calculations in the appendix) we have that the zero lower bound on the nominal interest rate becomes binding (for small numerical values of $\widehat{G}_L$) if and only if

$$\left(\phi_\pi\frac{\kappa}{1-\beta\mu} + \phi_y\right)\vartheta_r r_L + \overline{r} < 0. \tag{36}$$

Assuming that this condition holds (meaning that the financial disturbance is so large that $r_L$ becomes sufficiently negative), we have the zero lower bound is active whenever $\widehat{G}_L < \widehat{G}^{crit}$, where

$$\widehat{G}^{crit} \equiv \frac{\left(\phi_\pi\frac{\kappa}{1-\beta\mu} + \phi_y\right)\vartheta_r\left(-r_L\right) - \overline{r}}{\phi_\pi\frac{\kappa}{1-\beta\mu}\left(\vartheta_G - \Gamma\right) + \vartheta_G} > 0. \tag{37}$$

Then for $\widehat{G}_L < \widehat{G}^{crit}$ equations (33) and (31) become

$$\widehat{Y}_L = \vartheta_r r_L + \vartheta_G\widehat{G}_L \tag{38}$$

$$\pi_L = \frac{\kappa}{1-\beta\mu}\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] \tag{39}$$

for all $t > T$, which shows us that as long as the government is not able to increase its stimulus above the level $\widehat{G}^{crit}$, the economy will have a negative output gap and and there will be continuous deflation as long as the financial disruption exists. The government expenditure multiplier that is greater than one when the monetary policy is at the zero lower bound. This is due to the effect that the fiscal expansion causes the inflation to go up and thus the real rate to fall when the interest rate cannot be increased, as it would under an unconstrained Taylor rule.

A final note on the linearization applied here. We are merely using log-linear approximations around the zero-inflation steady state to the true structural equations derived earlier. This gives correct estimates of the multiplier only insofar as the deviations from this steady

[13] $\pi_t = \pi_L = \kappa\sum_{j=0}^{\infty}\beta^j E_t\left[\widehat{Y}_{t+j} - \Gamma\widehat{G}_{t+j}\right] = \kappa\sum_{j=0}^{\infty}\beta^j\mu^j\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] = \frac{\kappa}{1-\beta\mu}\widehat{Y}_L - \Gamma\widehat{G}_L$ for for all $t < T$. and $\widehat{Y}_L - \widehat{G}_L = \mu\left(\widehat{Y}_L - \widehat{G}_L\right) - \sigma\left(i_t - \mu\pi_L - r_t^{net}\right)$

state are small, but given that we consider the case where the disturbance is large enough to make the zero lower bound active, we should be cautious about immediately trusting the multipliers. Braun and Waki (2010) find that log-linearizing around the zero-inflation steady state can indeed exaggerate the size of the multiplier under realistic parameter values, but fortunately they find that this does not change the conclusion that the government expenditure is comfortably above 1.

### 3.1 Calibration of the model

We follow Eggertsson (2010)s(and Woodford (2010)) parameters that are chosen to match US data for the Great Depression. These parameter estimates for quarterly data can be found in the table below. In order to account for the large contraction during the Depression, the real interest rate was a little under minus 4 pct. p.a. and expected duration of the crisis was the little over 2.5 years.[14]

| | $\beta$ | $\kappa$ | $\sigma$ | $\Gamma$ | $r_L$ | $\mu$ | $\phi_\pi$ | $\phi_y$ |
|---|---|---|---|---|---|---|---|---|
| Value | 0.997 | 0.00859 | 0.862 | 0.425 | $-0.010$ | 0.903 | 1.5 | 0.25 |

(40)

For these values, the government expenditure multiplier is 2.29, which is considerably higher than the 1, which is the critical point of the on going debate.

## 4 Markov Model with Endogenous Switching

We now expand the Markov model with the feature that the value of fiscal stimulus will actually affect the probability of exiting the deflationary liquidity trap, so we expand our model with endogenous switching.

We assume once again that there is a significant level of financial disruption in the economy, but when the government decides a level of expenditure during the crisis $\widehat{G}_L$, this has a direct effect on the probability $\widetilde{\mu}$ that the disruption will remain in the following period (we call the probability $\widetilde{\mu}$ to make it distinct that this is the endogenous markov probability)

$$\widetilde{\mu} = \mu + \chi \widehat{G}_L, \tag{41}$$

so that in case government expenditure is not increased, the probability will be the same as in the exogenous markov case.

Now the Government expenditure is determined, which generates a markov process, and all household- firm- and monetary policy based equations are the same only with the exogenous probability $\mu$ replaced by the endogenous probability $\widetilde{\mu}$. By inserting equation (41) in (24), (29), and (25) we get the following three equations that make up the equilibrium in our endogenous model.

$$\left(1 - \beta\mu - \beta\chi\widehat{G}_L\right)\pi_L = \kappa\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] \tag{42}$$

$$\left(1 - \chi\widehat{G}_L\right)\left(\widehat{Y}_L - \widehat{G}_L\right) = -\sigma\left(i_L - \mu\pi_L - r_L\right) \tag{43}$$

$$i_t = \max\left\{0, \bar{r} + \phi_\pi\pi_t + \phi_y\left(\widehat{Y}_t - \Gamma\widehat{G}_t\right)\right\} \tag{44}$$

[14] These should not be microfounded rather found in surveys on macro data, since using microparameters would implicitly impose the very strong assumption that our model has Gorman aggregation.

Since we know this equilibrium for date $T$ and forward (this is unaffected by the endogenisation of the Markov process), we can solve for the equilibrium that persists while the interest rate spread is high. There exists a forward looking solution to the system of equation (24), (29), and (25) has a bounded solution if and only if the model parameters satisfy[15]

$$\sigma\widetilde{\mu}\kappa < (1-\widetilde{\mu})(1-\beta\widetilde{\mu}) \tag{45}$$

meaning that we have the level where $\kappa = (1-\theta)(1-\beta\theta)(\eta_u+\eta_v)/\theta > 0$. This inequality holds if $0 \leq \widetilde{\mu} < \overline{\mu} < 1$ for the upper bar $\overline{\mu}$, so we have the restriction on $\widehat{G}_L$ that

$$\chi\widehat{G}_L < \overline{\mu} - \mu. \tag{46}$$

When the nominal interest is zero we have that the solution to the model is

$$\widehat{Y}_L = \frac{\sigma\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)}{\beta\chi^2\widehat{G}_L^2-(1+\beta-\sigma\kappa)\chi\widehat{G}_L+(1-\beta\mu)(1-\mu)-\sigma\mu\kappa}r_L + \frac{\left[\beta\chi^2\widehat{G}_L^2-(1+\beta-\sigma\kappa\Gamma)\chi\widehat{G}_L+(1-\beta\mu)(1-\mu)-\sigma\mu\kappa\Gamma\right]\widehat{G}_L}{\left[\beta\chi^2\widehat{G}_L^2-(1+\beta-\sigma\kappa)\chi\widehat{G}_L+(1-\beta\mu)(1-\mu)-\sigma\mu\kappa\right]}$$

This relationship between $\widehat{Y}_L$ and $\widehat{G}_L$is illustrated graphically figure 1 below.

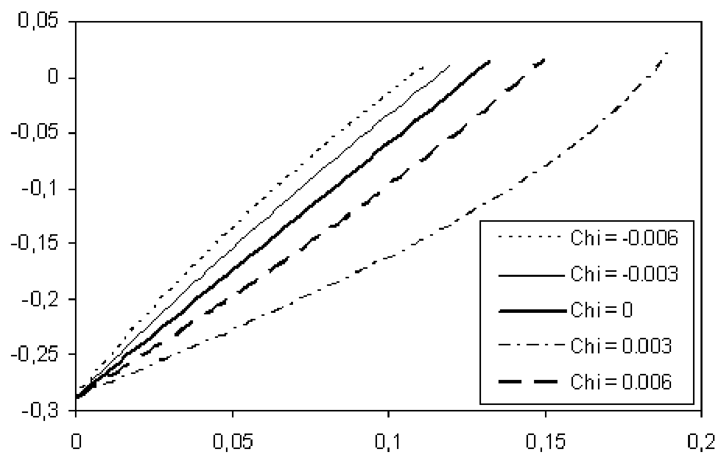



Figure 1:

(47)

All the lines have been cut off at the point, where the zero lower bound in the nominal interest rate became non-binding, and we see that the critical value of government expenditure is increasing in $\chi$, which is because a high positive level of $\chi$ would mean that the expected

[15] This condition will become obvious in equation XXX and XXX [the two V's)].

duration of the liquidity trap will decrease as government expenditure is increased, thereby causing the firms to expect a shorter period of deflation and hence decreasing deflation while in the trap and thereby further stimulating output.

The Government expenditure multiplier is of the form

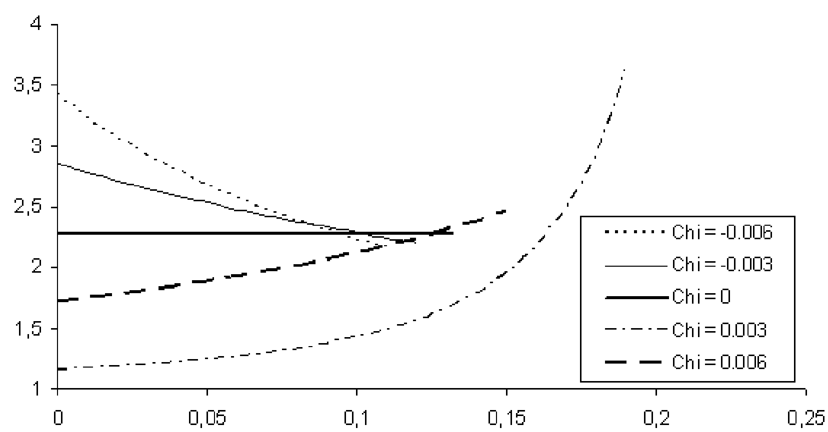
$$\frac{d\widehat{Y}_L}{d\widehat{G}_L} = A\left(\widehat{G}_L\right) r_L + B\left(\widehat{G}_L\right) \tag{48}$$

where

$$A\left(\widehat{G}_L\right) = -\frac{\sigma\beta\chi}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]} - \frac{\sigma\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\left[2\beta\chi^2\widehat{G}_L - \zeta\chi\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]^2} \tag{49}$$

$$\begin{aligned} B\left(\widehat{G}_L\right) = {} & \frac{\left[3\beta\chi^2\widehat{G}_L^2 - 2\left(1+\beta-\sigma\kappa\Gamma\right)\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\Gamma\right]\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]^2} \\ & - \frac{\left[\beta\chi^2\widehat{G}_L^3 - \left(1+\beta-\sigma\kappa\Gamma\right)\chi\widehat{G}_L^2 + \varpi\widehat{G}_L - \sigma\mu\kappa\Gamma\widehat{G}_L\right]\left[2\beta\chi^2\widehat{G}_L - \zeta\chi\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi\right]^2}. \end{aligned} \tag{50}$$

This multiplier is illustrated in Figure 2 below for different values of $\chi$




(51)

The multipliers are all above one, only when we set $\chi$as low as -0.006 are we beginning to get close to 1. The endogeneity of the markov process can have a very large effect on the government expenditure multiplier, even if the feedback parameter $\chi$ is only 0.003. The drop in expected duration lowers inflation, thereby increasing the real interest rate closer to zero and also increasing private demand due to less precautionary savings, means that the output effect of one marginal dollar of fiscal stimulus increases from 2.29 to 2.85.

# 5 Potential expansions

- We have assumed that the credit spread is zero in normal times. If we assume that there will be a positive small amount of credit friction in normal times, this will mean that the G-crit increases.

- Welfare effects can be found by comparing to the Market outcome in the perfect competition case, this is equivalent to the case where $\theta$ is 0 ($\epsilon$ should not matter because of the assumed subsidy) in our model.
- When interpreting these results we should keep in mind that we have assumed additive linear utility for the household, which leaves out the substitution effect. Woodford refers to Monacelli and Perotti (December 2010) for the effect of introducing non-separability.
- We assumed no capital.
- Assumed no liquidity constrained consumers (Barro) - such consumers will not have Ricardian equivalence and thus the timing of taxes would mater, at they would reduce consumption by exactly the tax when it is collected.
- Expanding analysis to a second order approximation, it would be more appropriate to use the more extensive second order setup.

# 6 Conclusion

The present paper has considered the Government expenditure multiplier in a New Keynesian DSGE model with an endogenous markov process. When the economy finds its monetary policy in a gridlock as the zero lower bound on the nominal interest rate becomes binding, there can potentially be scope for using fiscal stimulus as a means of substituting or assisting the monetary policy in lowering the negative output gap.

This paper found that when the economy is in a liquidity trap, the government expenditure multiplier is always greater than one. This is due to the effect that the fiscal expansion causes the real rate to fall when the interest rate cannot be increased, as it would under an unconstrained Taylor rule. For the parameters we chose in order to best approximate the economic conditions experienced during the Great depression, we found that the multiplier in the simple markov model was indeed well above one, it was 2.29.

We have argued that there are potential feedback mechanisms between the level of government expenditure during the crises and the probability that the economy will exit the liquidity trap. In the Endogenous Markov model we formalized this idea and found that insofar as fiscal stimulus will increase the persistence of the liquidity trap, the fiscal stimulus had a decreasing marginal effect on output, however the multiplier stayed above one. If the fiscal stimulus on the other hand reduced the expected duration of the crisis, the multiplier was significantly increased - to 2.85 and 3.42 for $\chi = -0.003$ and $\chi = -0.006$ respectively at $\widehat{G}_L = 0$. This indicates that the endogeneity of the markov switching probability can have potentially very large effects on the efficiency of fiscal policy when the economy is in a liquidity trap, and hence the topic should receive more attention in the future.

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# A Equilibrium in the Deterministic Model

We solve the system of linear equations by the method of undetermined coefficients. We conjecture a linear solution of the form

$$\widehat{Y}_t = \gamma_y \widehat{G}_t \tag{52}$$

$$\pi_t = \gamma_\pi \widehat{G}_t \tag{53}$$

$$i_t = \overline{r} + \gamma_i \widehat{G}_t \tag{54}$$

Inserting our conjectured solution form into the inflation rule, the Taylor rule and the Euler equation yields

$$\pi_t = \kappa \sum_{j=0}^{\infty} \beta^j E_t \left[\widehat{Y}_{t+j} - \Gamma \widehat{G}_{t+j}\right] = \kappa E_t \left[\left(\gamma_y - \Gamma\right) \widehat{G}_t\right] \sum_{j=0}^{\infty} (\beta\rho)^j = \frac{\kappa}{1-\beta\rho} \left(\gamma_y - \Gamma\right) \widehat{G}_t \tag{55}$$

Linearizing the monetary Taylor rule in and the Euler rule yields.

$$i_t = \max\left\{0, \overline{r} + \phi_\pi \pi_t + \phi_y \left(\widehat{Y}_t - \Gamma \widehat{G}_t\right)\right\} = \max\left\{0, \overline{r} + \phi_\pi \pi_t + \phi_y \left(\gamma_y - \Gamma\right) \widehat{G}_t\right\} \tag{56}$$

$$\widehat{Y}_t - \widehat{G}_t = E_t \left(\widehat{Y}_{t+1} - \widehat{G}_{t+1}\right) - \sigma \left(i_t - E_t \pi_{t+1} - \overline{r}\right) \tag{57}$$

Looking at the case when the interest rate is at the zero lower bound we then get the system of equations

$$\gamma_\pi \widehat{G}_t = \frac{\kappa}{1-\beta\rho}\left(\gamma_y - \Gamma\right)\widehat{G}_t \tag{58}$$

$$\gamma_i \widehat{G}_t = \phi_\pi \gamma_\pi \widehat{G}_t + \phi_y \left(\gamma_y - \Gamma\right)\widehat{G}_t \tag{59}$$

$$\left(\gamma_y - 1\right)\widehat{G}_t = \left(\gamma_y - 1\right)\rho\widehat{G}_t - \sigma\left(\gamma_i\widehat{G}_t - \gamma_\pi\rho\widehat{G}_t\right) \tag{60}$$

which can be reduced to

$$\gamma_\pi = \frac{\kappa}{1-\beta\rho}\left(\gamma_y - \Gamma\right) \tag{61}$$

$$\gamma_i = \phi_\pi\gamma_\pi + \phi_y\left(\gamma_y - \Gamma\right) \tag{62}$$

$$\gamma_i = \gamma_\pi\rho - \frac{1}{\sigma}\left(\gamma_y - 1\right)(1-\rho) \tag{63}$$

which can be solved for the three multipliers

$$\gamma_y = \frac{(1-\rho)+\psi\Gamma}{1-\rho+\psi} \tag{64}$$

$$\gamma_\pi = \frac{\kappa}{1-\beta\rho}\frac{(1-\rho)(1-\Gamma)}{1-\rho+\psi} \tag{65}$$

$$\gamma_i = \frac{(1-\rho)(1-\Gamma)}{1-\rho+\psi}\left(\frac{\rho\kappa}{1-\beta\rho} - \frac{\psi}{\sigma}\right) \tag{66}$$

where $\psi = \sigma\left[\phi_y + \frac{\kappa}{1-\beta\rho}(\phi_\pi - \rho)\right] > 0$. This shows that the government expenditure multiplier $\gamma_y$is necessarily larger than under perfectly flexible prices, as $\Gamma < \gamma_y < 1$.

# B Equilibrium in the standard Markov model

## B.1 Critical level of government expenditure

By substituting ) into () we get that

$$i_L = \max\left\{0, \overline{r} + \phi_\pi\pi_L + \phi_y\widehat{Y}_L\right\} \tag{67}$$

$$\widehat{Y}_L = \vartheta_r\left(r_L - i_L\right) + \vartheta_G\widehat{G}_L \tag{68}$$

$$\pi_L = \frac{\kappa}{1-\beta\mu}\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] = \frac{\kappa}{1-\beta\mu}\left[\vartheta_r\left(r_L - i_L\right) + \left(\vartheta_G - \Gamma\right)\widehat{G}_L\right] \tag{69}$$

$$i_L = \max\left\{0, \overline{r} + \left(\phi_\pi\frac{\kappa}{1-\beta\mu} + \phi_y\right)\vartheta_r\left(r_L - i_L\right) + \left[\phi_\pi\frac{\kappa}{1-\beta\mu}\left(\vartheta_G - \Gamma\right) + \vartheta_G\right]\widehat{G}_L\right\} \tag{70}$$

If we assume $\widehat{G}_L$ *is close to zero*, we get that
The zero bound on the nominal interest rate is thus binding if $\widehat{G}_L < \widehat{G}^{crit}$, where

$$\overline{r} + \left(\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y\right) \vartheta_r r_L + \left[\phi_\pi \frac{\kappa}{1-\beta\mu} (\vartheta_G - \Gamma) + \vartheta_G\right] \widehat{G}_L < 0 \tag{71}$$

$$\left[\phi_\pi \frac{\kappa}{1-\beta\mu} (\vartheta_G - \Gamma) + \vartheta_G\right] \widehat{G}_L < \left(\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y\right) \vartheta_r (-r_L) - \overline{r} \tag{72}$$

$$\widehat{G}_L < \frac{\left(\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y\right) \vartheta_r (-r_L) - \overline{r}}{\phi_\pi \frac{\kappa}{1-\beta\mu} (\vartheta_G - \Gamma) + \vartheta_G} \tag{73}$$

where we use that $\vartheta_G - \Gamma > 0$. This means that

$$\widehat{G}^{crit} \equiv \frac{\left(\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y\right) \vartheta_r (-r_L) - \overline{r}}{\phi_\pi \frac{\kappa}{1-\beta\mu} (\vartheta_G - \Gamma) + \vartheta_G} > 0 \tag{74}$$

if we assume that

$$\left(\phi_\pi \frac{\kappa}{1-\beta\mu} + \phi_y\right) \vartheta_r r_L + \overline{r} < 0. \tag{75}$$

Given our linearization, we note that on top of assuming $\widehat{G}_L < \widehat{G}^{crit}$, we also assume that $\widehat{G}_L$ is sufficiently close to zero.

# C Equilibrium in the endogenous Markov model

## C.1 Critical level of government expenditure

$$\begin{aligned}
(1-\beta\widetilde{\mu})\,\pi_L &= \kappa \left[\widehat{Y}_L - \Gamma \widehat{G}_L\right] && (76)\\
(1-\widetilde{\mu})\left(\widehat{Y}_L - \widehat{G}_L\right) &= -\sigma\left(i_L - \widetilde{\mu}\pi_L - r_L\right) && (77)\\
i_t &= \max\left\{0, \overline{r} + \phi_\pi \pi_L + \phi_y \left(\widehat{Y}_t - \Gamma \widehat{G}_t\right)\right\} && (78)
\end{aligned}$$

which by substitution yields

$$-\sigma\left((1-\beta\widetilde{\mu})\, i_L - \widetilde{\mu}\kappa\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] - (1-\beta\widetilde{\mu})\, r_L\right) = (1-\beta\widetilde{\mu})(1-\widetilde{\mu})\left(\widehat{Y}_L - \widehat{G}_L\right) \Leftrightarrow$$

$$\sigma(1-\beta\widetilde{\mu})(r_{L-}i_L) = \left[(1-\beta\widetilde{\mu})(1-\widetilde{\mu}) - \sigma\widetilde{\mu}\kappa\right]\widehat{Y}_L - \left[(1-\beta\widetilde{\mu})(1-\widetilde{\mu}) - \sigma\widetilde{\mu}\kappa\Gamma\right]\widehat{G}_L \Leftrightarrow \tag{79}$$

Insert $\widetilde{\mu}$ and assume that we are at the zero lower bound, $i_L = 0$

$$\widehat{Y}_L = \frac{\sigma\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)}{\beta\chi^2\widehat{G}_L^2 - (1+\beta-\sigma\kappa)\chi\widehat{G}_L + (1-\beta\mu)(1-\mu) - \sigma\mu\kappa} r_L + \frac{\left[\beta\chi^2\widehat{G}_L^2 - (1+\beta-\sigma\kappa\Gamma)\chi\widehat{G}_L + (1-\beta\mu)(1-\mu) - \sigma\mu\kappa\Gamma\right]\widehat{G}_L}{\left[\beta\chi^2\widehat{G}_L^2 - (1+\beta-\sigma\kappa)\chi\widehat{G}_L + (1-\beta\mu)(1-\mu) - \sigma\mu\kappa\right]}$$

We further use the polynomial notation for equation (**??**)

$$H\left(\widehat{G}_L\right)*\widehat{Y}_L - F\left(\widehat{G}_L\right) = J\left(\widehat{G}_L\right) r_L \tag{80}$$

where F, G and H are 3rd, 2nd and 1st order polynomials respectively. ,

$$\widehat{Y}_L = J\left(\widehat{G}_L\right) H^{-1}\left(\widehat{G}_L\right) r_L + F\left(\widehat{G}_L\right) H^{-1}\left(\widehat{G}_L\right) \tag{81}$$

The isolation of $\widehat{Y}_L$ only holds if $H\left(\widehat{G}_L\right) \neq 0$.Then the derivative is

$$\begin{aligned}\frac{d\widehat{Y}_L}{d\widehat{G}_L} &= \frac{\frac{dJ\left(\widehat{G}_L\right)}{d\widehat{G}_L} H\left(\widehat{G}_L\right) - J\left(\widehat{G}_L\right)\frac{dH\left(\widehat{G}_L\right)}{d\widehat{G}_L}}{H^2\left(\widehat{G}_L\right)} r_L + \frac{\frac{dF\left(\widehat{G}_L\right)}{d\widehat{G}_L} H\left(\widehat{G}_L\right) - F\left(\widehat{G}_L\right)\frac{dH\left(\widehat{G}_L\right)}{d\widehat{G}_L}}{H^2\left(\widehat{G}_L\right)} \\ &= A\left(\widehat{G}_L\right) r_L + B\left(\widehat{G}_L\right) \end{aligned} \tag{82}$$

then

$$A\left(\widehat{G}_L\right) = -\frac{\sigma\beta\chi}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]} - \frac{\sigma\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\left[2\beta\chi^2\widehat{G}_L - \zeta\chi\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]^2} \tag{83}$$

$$\begin{aligned}B\left(\widehat{G}_L\right) = {} & \frac{\left[3\beta\chi^2\widehat{G}_L^2 - 2\left(1+\beta-\sigma\kappa\Gamma\right)\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\Gamma\right]\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi - \sigma\mu\kappa\right]^2} \\ & - \frac{\left[\beta\chi^2\widehat{G}_L^3 - \left(1+\beta-\sigma\kappa\Gamma\right)\chi\widehat{G}_L^2 + \varpi\widehat{G}_L - \sigma\mu\kappa\Gamma\widehat{G}_L\right]\left[2\beta\chi^2\widehat{G}_L - \zeta\chi\right]}{\left[\beta\chi^2\widehat{G}_L^2 - \zeta\chi\widehat{G}_L + \varpi\right]^2}\end{aligned} \tag{84}$$

where $\varpi \equiv (1-\beta\mu)(1-\mu)$ and $\zeta \equiv 1+\beta-\sigma\kappa$.
Condition for being in the liquidity trap

$$i_L = 0 \Leftrightarrow \tag{85}$$

$$\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\overline{r} + \phi_\pi\kappa\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] + \phi_y\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\left(\widehat{Y}_t - \Gamma\widehat{G}_t\right) < 0 \Leftrightarrow \tag{86}$$

$$\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\overline{r} + \phi_\pi\kappa\left[\widehat{Y}_L - \Gamma\widehat{G}_L\right] + \phi_y\left(1-\beta\mu-\beta\chi\widehat{G}_L\right)\left(\widehat{Y}_t - \Gamma\widehat{G}_t\right) < 0 \tag{87}$$