Reserve Price in Search Models - *

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1 Introduction

Search models is an extensive and important field, with a large literature. The field, originally developed due to diamond paradox [4] had since grown and covers many topics, such as consumer search, for example [9], [10], [2] and many other fields. One of the most popular models applied in the search literature is a model introduced by Stahl in [9]. It describes a simple consumer search with two types of consumers, both with a reserve price. Additional research, for example, [5], show that Stahl model performs very well in reality and predicts pricing of goods very well. Therefore, this paper concentrates on the Stahl model.

Recently there develops literature on an extension of the Stahl model, where the sellers are heterogeneous, for example [1], [3], [7]. One of the results of such models is that a symmetric NE is no longer possible when the sellers are heterogeneous. This is a very important assumption, as in reality sellers vary in size and popularity. Therefore, it is important to note that asymmetric NE play an important role in this extension. However, once asymmetric NE arise, it creates an additional complexity. Throughout the search literature there is a crucial assumption of a searcher reserve price. This allows easier analysis of the game and is used in proving many of the results. However, as noted in [8] once an asymmetric strategies are in place, additional NE, without a consumer reserve price, may exist. The reason behind it is simple - once a price was revealed to the searcher, she can interpret this price and gain information regarding prices in other stores. The information comes from the Bayesean beliefs of the searcher regarding the strategy that

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was realized at a given store, leading to beliefs regarding the strategies used by other sellers.

1.1 Contribution

This paper provides a condition on the game which ensures existence of a reserve price. The result has a very important application allowing the search literature to remain in the reserve price world, once this condition is met. The condition is on beliefs of consumers. Their main property is allowing free undercut for sellers, namely, at certain prices it is possible to offer arbitrarily small discounts without losing consumers. Therefore the condition on the beliefs is called 'undercut proof'.

Similar result can hold for many other search models, which have a similar problem. The relevant models are with a finite number of heterogeneous sellers (or firms offering jobs). There asymmetries can lead to asymmetric choice of strategies and to additional equilibria, without reserve price.

The structure of the paper is as follows: first the Stahl model is introduced. Then the problem of no reserve price is illustrated in an example. Afterwords, 'undercut proof' condition is introduced and lastly, it is shown that such a condition is a sufficient one for reserve price to exist in NE.

2 Model

The model is similar to the Stahl model . The main difference is the possibility for heterogeneity among sellers.

In the model there are S sellers, selling an identical good. Each seller owns n_i of the total N stores. The production cost is normalized to 0, and seller can meet demand. Additionally, there are buyers, each of whom wishes to buy a unit of the good. The mass of buyers is normalized to 1. This implies that there are many small, strategically insignificant buyers.

The sellers are identical, and set their price once at the first stage of the game. The same price is set for all the stores of a given seller. If the seller mixes then the distribution is selected simultaneously, and only at a later stage the realizations take place. Note that n_i are common knowledge among sellers.

The buyers are of two types. A fraction μ of buyers are shoppers, who know where the cheapest price is, and buy there. In case of a draw they randomize over all cheapest stores, spreading equally among them. The rest are searchers, who sample prices sequentially. Sampling price in the first, randomly and uniformly selected, **store** is free. It is shown in [6] that if the first visit is costly some searchers would avoid purchase and the rest will take the first search cost as given. If the price there is satisfactory - the searcher will buy there. However, if the price is not satisfactory - the searcher will go on to search sequentially in additional stores. Each additional search, ending with observing a price at an additional store, has a cost c. The second (or any later) store is randomly and uniformly selected from stores owned by previously unvisited sellers. The searcher may be satisfied, or search further on. When a searcher is satisfied, she has a perfect and free recall. This implies she will buy the item at the cheapest store she had encountered, randomizing **uniformly** in case of a draw.

Before going on, some technical assumptions on the model are introduced, and the rationality behind them is explained. The assumptions are as follows:

- Throughout this paper it is assumed that the sellers cannot offer a price above some finite bound M. This has the interpretation of being the maximal valuation of a buyer for the good.
- Throughout this paper, it is also assumed that searchers accept any price below c. The logic behind it is any price below my further search cost will be accepted, as it is not possible to reduce the cost by searching further.
- To avoid measure theory problems it is assumed that mixing is possible by setting mass points or by selecting distribution over full measure dense subsets of intervals.

2.1 Knowledge and Beliefs

Searchers have beliefs regarding the prices sellers set. For each possible (pure or mixed) strategy s of the model is attached a belief, stating how many stores are priced according to strategy $n(s)$ (clearly the sum of $n(s)$ is n, the number of stores). Each strategy has an expected price, denoted $e(s)$. Searchers do not know or observe the number of stores each seller has, but form beliefs according to the available sizes of stores.

Suppose the searcher observed the price p. Let the probability that this price p came from strategy s be denoted as $prob(p, s)$. To calculate $prob(p, s)$ first calculate chance that s is selected by some seller, according to searchers beliefs. Additionally the probability that p is the realization of strategy s (relevant for mixed strategies) is weighted in. One needs to note that if some strategies (with positive $n(s)$) have a mass point on p only those will be considered, otherwise the densities will play a role. Formally:

$$prob(p,s) = \frac{n(s)f(s)}{\sum_{p\in s'} n(s')f(s')} \tag{1}$$

Now, if the searcher thinks that strategy s was selected, searching further will yield (in expected terms) the expected price in other sellers stores. Namely, it is the expected price, only that $n(s)$ is now lower (as s was observed in one of the stores). Let $E(n_i)$ denote the expected size of a seller using strategy s. Then the calculation is as follows (clearly if $E(n_i) \geq n(s)$ the difference will be zero and the element omitted):

$$\frac{\sum_{s':n(s')>0,s'\neq s} n(s')e(s') + (n(s) - E(n_i))^{+}e(s)}{\sum_{s'} n(s')} \tag{2}$$

Definition 2.1 *As the beliefs are on the sellers disregarding their identity, we will refer to this belief setting as 'Anonymous Knowledge'.*

2.2 No Reserve Price Example

Searchers search further only when the expected price in a search is at least c lower than the lowest price observed so far. Below is an example of how to calculate an expected search price, and additionally illustrates that no reserve price may exist:

Example 2.1 *Suppose the search cost c is 0.9 and pricing strategies, equally probable from searchers beliefs, are as follows:*

1. *Uniform in [1, 9], Exp. value of 5*
2. *Uniform in [5, 9], Exp. value of 7*
3. *Pure strategy of 7.*

After observing the price of 7 the searcher is certain with prob. 1 that she had encountered the third strategy seller. An additional search will yield the average between the expected values of the two strategies: 6, making an additional search worthy.

After observing the price of 7.1 the searcher knows that she had encountered one of the mixed strategies, and due to a likelihood ratio - twice more probable that it is the second strategy. Therefore, with probability 1/3 it is the first str. and probability 2/3 the second str.

If the first strategy was encountered, then an additional search will end up in ether second or third strategy - both with expected price of 7.

If it is the second strategy, then an additional search will end up with expected price of 5 or of 7, as both can occur with equal probability (due to the beliefs) expected price in an additional search in this case is 6.

Combining the two possibilities, when taking into account that the second case is twice more probable, the expected price in an additional search is $(2 \cdot 6 + 7)/3 = 6.333$, making another search not profitable.

Here one sees the problematic assumption of the reserve price - it might be the case that it does not exist. However, the literature concentrates on NE with a reserve price. This paper will provide a basic condition for existence of a reserve price, which will involve a specific type of trembling hand perfection.

2.3 Satisfaction Sets

Searchers in the course of a search observe prices, and after some price vectors observed the search ends. Let the search ending vector sets be defined as buyer satisfaction sets. Formally, a satisfaction set BS_k consists of all vectors of length k, such that after observing the k prices denoted in the vector, in the corresponding order, the searcher is satisfied, and does not search further. Note that since any price below c is satisfactory BS_k is non empty for all k.

Let v be a vector in BS_k for some $k \geq 1$. The lowest coordinate value in v is denoted as v_m. The set of coordinates in v with v_m will be denoted $Min(v)$. From the definitions v_m is the price paid by the searcher after visiting k stores and observing the prices vector v.

For example, a vector $v = \{8, 5, 9\} \in BS_3$, has $v_m = 5$. v implies the following: the searcher visited 3 stores. In the first observed the unsatisfactory price price of 8. In the second the price of 5 was observed, and still was unsatisfactory. After a third search the price of 9 was observed, making the price 5 satisfactory. Such an example can arise when the last visited seller mixes between 9 and some extremely low prices. After seeing that the realization was 9, a previously unsatisfactory price becomes attractive.

Let the supremum element of BS_1 (the first satisfaction set) be denoted as P_M. Let P_S denote the maximal (supremum) price which can be set by a seller. Implying - in NE no one offers a price above P_S. From an assumption no price above M can be set, and therefore, P_S and P_M are well defined and finite.

Note the following property. Let P_{min} be the lowest (infimum) price that can be offered in

NE. Any price below P_{min} is in BS_1. Namely, any price below the lowest price offered in NE is satisfactory. This follows directly from the Bayesian structure of the beliefs: such price is satisfactory as any additional search will end up with a weakly higher price.

Lastly, an undercutting condition is defined, which will lead to existence of a reserve price.

Definition 2.2 *Let σ be a strategy profile of the sellers. Let us denote by MP the set of prices where some sellers have set a mass point.*

Definition 2.3 *Let v be a price vector in BS_k with $v_m = p$. Suppose exists $\varepsilon > 0$ such that when subtracting any number smaller than ε from any single coordinate in $Min(v)$ will keep the vector in BS_k. Then v is denoted ε-undercut proof.*

Definition 2.4 *Fix a natural number k. Suppose that for any price p in MP exists $\varepsilon_p > 0$ such that for any vector v of length k with $v_m = p$, v is ε_p-undercut proof. Then BS_k will be denoted as undercut proof.*

The condition says the following: if a certain vector is satisfactory, then reducing one of the lowest elements in it by a small amount will not affect the decision of the searchers. The main implication of this assumption is simple - for any price p it is possible to undercut it without losing any searchers. This condition can be interpret as a trebling hand perfection condition. A seller sets a certain price with a mass point, but due to a trembling hand the price is set slightly lower. As the number of mass points with mass above any positive number is finite, it is possible to take an interval without any significant mass points. The searcher observing such a price will think that it is much more probable that a seller trembled rather than such a price was selected by a distribution without mass points.

2.4 Game Structure

The game is played between sellers, searchers and shoppers. The time line of the game is as follows:

1. Sellers select pricing strategies
2. Realizations of prices occur for sellers with mixed strategies.
3. Shoppers go and purchase the item at the cheapest store
4. Searchers select a store and observe the price in the store

5. If the price observed is in BS_1 the searcher is satisfied and purchases the item, if not the search continues
6. All unsatisfied searches select one additional store, pay c and sample the price there.
7. If the price vector observed is in the corresponding satisfaction set the searcher is satisfied and purchases the item, if not the search continues
8. ...
9. When the seller observed all stores and observed only prices above P_M she would buy at the cheapest store encountered.

- Searchers have beliefs about which strategies were actually played by the sellers and about sellers sizes.
- Shoppers will know the real price in each store in the moment it is realized.

The probability that seller i sells to the shoppers when offering price p is denoted $\alpha_i(p)$. Let q be defined as the expected quantity that seller i sells when offering price p. It consists of the expected share of searchers that will purchase at her store, plus the probability she is the cheapest store times μ. It is is also the market share of the seller.

The utilities of the game are as follows:

- The seller utility is the price charged multiplied by the expected quantity sold.
- The consumer utility is a large constant M, from which the price paid for the item and the search costs are subtracted.

The NE of the game, under our assumptions, is as follows:

- The searchers beliefs coincide with the actual strategies played.
- The searching strategy rational for the searchers, according to 'Anonymous Knowledge'.
- Sellers know that searchers are rational.
- No seller can unilaterally adjust the pricing strategy and gain profit in expected terms.

Remark 2.1 *As the sum of the searcher and seller utilities may differ only in the search cost, any strategy profile where the searchers always purchase the item at the first store visited is socially optimal.*

3 Results

The main result of the paper is as follows:

Theorem 1 *Suppose that all BS_k are undercut proof. Then all NE of the model have reserve price of P_M, up to zero measure adjustments. Additionally, P_M is the only price which may have mass points in NE.*

Firstly, let me address the zero measure adjustments. It might be the case that all prices below some P_{res} are accepted, except several prices of measure zero. It is possible to exclude those points from any distribution function without significantly changing it, and these are the only changes allowed by the theorem. In combination with the no mass points, it is clear that such prices will affect the equilibrium with probability 0, as the probability for such 'unsupported' price to be selected is zero.

For example, consider a continuous distribution on $[a, b]$ with positive density everywhere in the interval. Now, if $c \in [a, b]$ is excluded from the distribution, the probability for the change to make a difference is zero.

The theorem will be proved in a sequence of lemmas. Firstly, it is shown that no price above P_M is used and P_M cannot be an isolated point of BS_1. Following step is showing that no mass point exist, except possibly at P_M, and then that no interval holes can exist in equilibrium. All lemmas refer to a NE strategy profile of the game.

Lemma 3.1 *If $P_S > P_M$ there are no mass point at P_S.*

Suppose that a single seller sets a mass point at P_S. As this price is not satisfactory for searchers at their first visit, searchers will not buy at this price and search further. All prices in all other stores are below P_S w.p. 1, and therefore, none of them will ever return. From the same reason no other searcher or any shoppers will buy at the price of P_S. Thus, P_S yields the profit of 0, and a profitable deviation would be to select c instead of P_S.

Examine the case that several sellers set a mass point at P_S. The price might be satisfactory for searchers, who search in k stores and observe only P_S. Suppose $(P_S, P_S, ...P_S)$ of length k is in BS_k. A profitable deviation in this case would be to undercut the price, as due to the undercut proof condition the searchers will still be satisfied. $\square$

Corollary 3.1 *In NE, for any price p with positive prob. to attract shoppers, at most a single seller has a mass point at p.*

If several sellers have a mass point at some price p, there is a positive prob. for multiple sellers to select p. Then undercutting is possible and the shoppers share will increase discontinuously. Due to the undercutting condition no searchers will be lost. The prob. to satisfy shoppers is weakly higher. The result is that undercutting is a profitable deviation.

Lemma 3.2 *No price above P_M is selected.*

After lemma 3.1 all remaining to show is that if $P_S > P_M$, then P_S cannot be selected without mass points. If it is the case, the profit of selecting P_S is arbitrarily close to 0, as the probability for a different seller to select a price above is arbitrarily close to zero. At some stage, a deviation to c would be profitable. $\square$

Lemma 3.3 *If $P_M \in BS_1$ it is not an isolated point of BS_1.*

Suppose P_M is isolated point of BS_1. From the undercut condition this can occur only if no seller set mass points on P_M.

Note that if P_M is an isolated point of BS_1 and a seller selects it, then such seller must also select a price arbitrarily close to P_M. Such prices would be outside BS_1. Note that from lemma 3.2 no price above P_M is ever selected, and with probability arbitrarily close to 1 searchers would observe lower prices in future searches. From here it is clear that such prices will yield a profit arbitrarily close to 0, and therefore will not be selected. Therefore, if P_M is an isolated point of BS_1 and in support of sellers it must be done so with a mass point. Thus, if P_M is isolated in BS_1 no seller can have P_M in support.

Suppose that P_M is not selected by any seller, and the highest (supremum) price in support is some $p < P_M$. If the price has positive prob. to attract shoppers then all sellers must have a mass point at p, contradicting corollary 3.1. On the other hand, if such price has zero prob.[1] to attract shoppers. Then a deviation to a higher price p' in BS would be profitable. With p' all the initial searchers buy at the store, and the loss of shoppers is arbitrarily small. Since $p' > p$ the profit will be higher. Therefore, p' must be in the support of at least one seller. Note that when offering p (or arbitrarily close prices to p) the searchers share is at most the initial visiting searchers, as is when offering p'. Thus a contradiction is reached. $\square$

From the construction, BS_1 is dense in some interval of the form $(a, P_M]$. Let us define the highest interval in BS_1 as follows:

Definition 3.1 *Let I be the interval of the form $(a, b]$ such that $b = P_M$ and a is the highest number below P_M such that exists ε satisfying: $(a - \varepsilon, a) \cap BS = \emptyset$.*

[1]or arbitrarily close prices below p have arbitrarily close to zero prob.

This allows I to cover some prices that are not covered in BS_1, as noted in the theorem. Besides, I is the highest interval of the first satisfaction set. Additionally note that the definition does not imply anything on whether P_M is or is not in BS_1.

Lemma 3.4 *Prices in $I \setminus BS$ have no density*

Suppose a price $p \in I \setminus BS$ has some density by seller i. From definition p is not the infimum of I, and the arbitrarily close prices below p are in BS_1. Note that there is a positive probability for some sellers to select prices below p, and therefore, some of the searchers initially visiting i will not purchase there.

Thus, a deviation to a price just below p, which is in BS_1 from construction, would be a profitable deviation. The loss in price is arbitrarily small. Increase in searchers purchasing at seller i would overweight it. $\square$

Let the maximal (supremum) price in the sellers support which is not in BS_1 be denoted as P_{src}. Additionally, let the maximal (supremum) price which has a positive probability to attract shoppers be denoted as P_{shop}. The following statements deal with the relationship between these two prices:

Remark 3.1 *Note that from construction P_{src} is not the lowest price which can be offered in NE. From here follows that with positive prob. some of the searchers observing the P_{src} at their first visit*[2] *will purchase at a lower price. Additionally, all searchers*[3] *who were not satisfied after their first visit observed a lower price. Therefore, when P_{src}*[4] *is offered the share of searchers supplied is strictly lower than when all initial searchers visiting the store buy there.*

Lemma 3.5 *$P_{shop} = P_M$ In words: P_M is the supremum point of strategy support for each seller.*

From lemma 3.2 it cannot be higher than P_M.

Suppose that the supremum price of seller i is $p < P_M$, an is the lowest supremum among all sellers. Formally, $P_{shop} = p < P_M$. For any price above p and below P_M the probability to sell to shoppers is 0.

Suppose $P_{src} \in (p, P_M) \setminus BS$. As noted above, seller setting the price of P_{src}[5] would receive a share of searchers strictly lower than the searchers initially visiting her store. Therefore,

[2] or prices arbitrarily close below and outside BS_1 in the supremum case

[3] or arbitrarily close to all searchers in supremum case

[4] or arbitrarily close prices below it outside BS_1 in supremum case

[5] or arbitrarily close below it if P_{src} is the supremum

a profitable deviation would be to select a price arbitrarily close to P_M (which is in BS_1 according to definition or undercutting condition). The prob. to sell to shoppers in both cases is zero, but the share of searchers and the price are higher in the case of a price just below P_M. Similarly, if a seller selects a price in BS_1 and in (p, P_M), a profitable deviation would be to select a higher price in $(p, P_M) \cap BS$. Therefore, no seller selects prices in (p, P_M).

If there are no mass points at p, seller i can deviate to a price just below P_M instead of p an gain profit due to a higher price. If a single seller j (can also be i) has a mass point at p then j can deviate to a price just below P_M.

Due to corollary 3.1 the only case remaining is when several sellers have a mass point at p, but seller i is not one of them. In such case any of those sellers can deviate profitably to P_M. To conclude, there is a profitable deviation for at least one seller in all cases. Therefore, in NE it is not possible for a seller to have a support supremum below P_M. □

Corollary 3.2 *Let p be a price below P_M. At most a single seller has a mass point at p.*

Follows directly from Lemma 3.5 and Corollary 3.1.

Lemma 3.6 *If there are mass points in I, these must be at P_M.*

Suppose that exist some mass points in I at a price $p < P_M$. Due to corollary 3.2, at most a single seller has a mass point at p. Let i denote this seller. Due to lemma 3.4 all searchers observing a price above p are satisfied after first visit.

Note that if there is a mass point at p set by seller i. the probability to attract shoppers decrease discontinuously at that price. Therefore, if any seller except for i would not select prices just above p for all sellers except i. As a result i can deviate and increase the price from p to a slightly higher price. The probability to attract shoppers remains the same, no searchers are lost as only the searchers visiting the store initially are relevant, and all those already purchase the item as the price is in BS_1. Therefore, if such mass point exists, a profitable deviation exist to some sellers. □

Now let us examine the prices strategies below the interval I. Let us denote the maximal (supremum) price that can be offered by sellers below I as P_I. Note that if $P_I \notin BS_1$ then P_I is not the lowest price offered in NE. Let us examine whether P_I is in BS_1 or outside it.

Lemma 3.7 *P_I must be in BS_1*

Note that no seller selects a price above P_I and below I. From corollary 3.2 at most a single seller has a mass point at P_I. There are two possibilities: ether some seller i has a mass point at P_I or no seller has a mass point there.

In the first case seller i can deviate into the interval I, arbitrarily close to the infimum of I, instead of selecting P_I. The loss of shoppers is arbitrarily small, as no other seller has any mass on any prices between P_I and I, and the only possible mass points in I are at P_M. The price is higher, and the share of searchers will be weakly higher. The latter is due to the fact that all initial searchers now purchase at seller i, and previously some could observe a lower price and never return. Note that due to lemma 3.4 all unsatisfied searchers after first visit observe a price of at most P_I, implying that additional unsatisfied searchers will not purchase at i.

If there are no mass points at P_I due to similar reasoning any seller who has P_I in her support can deviate from P_I (or prices sufficiently close to it below P_I in supremum case) into I. Thus, in any case when P_I is not in BS_1 a profitable deviation exists. $\square$

Lemma 3.8 *P_I cannot be in BS_1*

From corollary 3.2 at most a single seller has a mass point at P_I. Suppose seller i has a mass point at P_I, and no other seller has a mass point at I. Deviating into I, arbitrarily close to the low end of I, is a profitable deviation:

Searchers fraction will remain the same. Due to lemma 3.6 there are no mass points in I except at P_M. Additionally, no seller has a mass point at P_I. This implies that no seller except i selects prices weakly above P_I and below I. From here follows that the probability to attract shoppers is arbitrarily close to the one at P_I. Combining it all, the price charged will be higher and the quantity sold will be arbitrarily close, yielding a profitable deviation.

If there are no mass points at P_I the above describe deviation from P_I or prices arbitrarily close to it, is still profitable for the same reasons. $\square$

Combining the lemmas we see that no price below I will be offered in NE, and therefore any price below I would be in BS_1. This completes the proof of the theorem.

4 Summary

The result presented here allows to extend the consumer search models and allow dealing with a finite number of hetrogeneous sellers. There some NE without a reserve price may exist, but in very limited setting. The condition presented here allows to eliminate the possibility of NE without a reserve price, allowing a simple analisys of many search models.

Further research can deal with additional search models in various field, labor search for example. Testing whether such a condition is a sufficient one too. This will allow to make the search models more extended and realistic. An additional further research is to identify such NE without a reserve price and find a characterizing condition on when those are possible, if at all.

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