Coalitional stochastic stability

Jonathan Newton *

August 26, 2008

Abstract

In this paper we define and discuss an equilibrium selection criterion which we call *coalitional stochastic stability*. This differs from the existing work on stochastic stability in that profitable coalitional deviations are given greater importance than unprofitable single player deviations. Coalitionally stochastically stable states are shown to exist and an intuitive and simple method is given for finding them. This method is shown to give a simple lexicographic ranking of efficiency and risk-dominance for 2-player games. Coalitional stochastic stability is also shown to select more plausible equilibria in games of contracting than standard uniform error models of stochastic stability. Our approach also provides a novel justification for interpretations of random errors as experimentation which currently exist in the literature. We also discuss how one of the problems affecting stochastic stability methods - large expected time to convergence to stable states - can be mitigated using a coalitional approach. Finally, this paper is related to work on the noncooperative foundations of cooperative game theory and raises challenges to cooperative equilibrium concepts from an evolutionary perspective.

Keywords: Stochastic stability, learning, coalition, lexicographic, contract, experimentation, time to convergence.

JEL classifications: C71, C72, D83

*Department of Economics, University of Cambridge. This paper is solely the work of Jonathan Newton. Please do not quote from this paper without permission of the author, which can be obtained by emailing jcsn2@cam.ac.uk

1 Introduction

In his seminal paper in evolutionary game theory Young (1993) introduces the idea of *stochastic stability*: a method of equilibrium selection in settings featuring adaptive learning. Players repeatedly play an n player game Γ. Players follow a process whereby they play best responses to a distribution over the actions played by the other players, where the distribution is determined by sampling s actions from the previous m actions played by the other players. This defines a Markov process where the states of the process are defined by the actions taken by each of the players in the previous m periods. If s is small enough relative to m and the game is *weakly acyclic* the game converges almost surely to a *convention* where each player has played a strategy from a pure strategy Nash equilibrium of Γ for as long as anybody can remember. Young introduces random shocks to the system which can be interpreted as random mistakes made by players in implementing their strategies. As long as there is a positive probability of every combination of strategies being played by mistake the perturbed system then defines an aperiodic and irreducible Markov process. Young shows that as the probability of a random shock ϵ approaches zero, the system spends almost all of its time in a subset of the conventions of the games. He calls such conventions *stochastically stable*.

Here I argue that stochastic stability can lead to unrealistic results in games where coalitional behaviour can be expected by players. Furthermore I define *coalitional stochastic stability* which I claim can lead to a more realistic equilibrium selection in certain settings, particularly in games with more than 2 players.

2 Related literature

There are two strands of literature which this paper bridges. Below I give a brief summary of both of them followed by a description of the motivation and contribution of this paper.

2.1 Coalitional behaviour

There exists a gigantic literature in cooperative game theory on the behaviour of coalitions. For a survey the reader is referred to Peleg & Sudholter (2003). Aumann (1959) gives the concept of a 'strong equilibrium' - an equilibrium where no subset of players would want agree to change their profile of strategies to another profile, holding the strategies of all players not in that subset fixed. This equilibrium concept can be argued to correspond to situations where coordination between any subset of players is possible without players outside the subset being aware of it. As the name suggests this is a very strong equilibrium notion and often will not exist. The concept can be weakened to that of k-strong equilibrium where only coalitions of size k or lower have to have their incentive constraints satisfied but still existence is not guaranteed [1]. Bernheim *et al.* (1987) attempt to address the issue of robustness to coalitional deviations through their concept of coalition proof equilibrium, the idea of which is that equilibria need to be robust to a set of players deviating only if that set of players is itself robust to any further deviations by subsets of its constituent players. Bernheim *et al.* argue that this equilibrium concept can be understood intuitively to lead to outcomes which could be reached if all players seated in a room reached an agreement, following which the players leave the room one by one, and no matter in what order they leave the room there will never be a subset of players remaining in the room who would agree to play differently to what was agreed with all players present. Konishi & Ray (2003) look at the issue of coalition formation in a dynamic setting with farsighted agents, showing that if characteristic functions are incorporated into the rules governing the dynamic process then they can choose such a process which always selects payoffs in the core of the underlying game if the core is a singleton.

2.2 Stochastic stability

Young (1993) and Kandori *et al.* (1993) introduce the idea that although there may be several stationary states in a dynamic process, some of them may be more robust to 'random errors' in strategies chosen than others, and that if the probability of a random error in the very long run becomes very small, then the state which is most robust to such errors will be observed almost all of the time. Young (1993) predicts that in 2x2 games when there are two strict Nash equilibria then the risk dominant equilibrium will be selected. It has been noted by Ellison (1993) that the time taken for random errors to cause a switch between stationary states of the underlying process can be incredibly long, although Ellison (2000) and Naidu & Bowles (2005) note that if movement between states of the underlying process primarily takes place between states which are 'close' to one another then the time required for switches can be lower. Bergin & Lipman (1996) prove a kind of folk theorem for stochastic stability, that is they show that any stable state of the unperturbed dynamic process can be selected with appropriately chosen

[1] Nash equilibrium is a special case of k-strong equilibrium where k=1

state-dependent mutation rates. van Damme & Weibull (2002) recover some of the predictive power of the theory by assuming that not making mistakes is costly to players. They endogenize the random error probabilities so that players will pay more to avoid making mistakes when it is more costly to them and give conditions under which random errors between players and for a given player in different states remain of the same order of magnitude as limits are taken. They show that under these conditions the results of Young are recovered. Young (1998a) shows that under his uniform error stochastic stability process there is a preference for fairness in contracts agreed to between two players and that the contract selected corresponds to the Kalai-Smorodinsky bargaining solution. Naidu & Bowles (2005) analyse a model of contracting where movement from one Pareto efficient contract to another is only caused by errors on the part of the player who stands to gain from the move. They justify this by invoking a level of foresight on the part of the agents, who know that there is a better contract available and thus 'intentionally' make mistakes so as to lead to a better conventional contract for themselves.

2.3 Motivation and contribution of this paper

2.3.1 Individual rationality

Much has been written on equilibrium selection by perturbing the 'rationality' part of 'individual rationality'. Hardly anything has been written regarding perturbing the 'individual' part. It is not unreasonable to expect that players may occasionally take irrational actions, but neither is it unreasonable to think that from time to time players may meet and agree to jointly coordinate their actions. It is also clear that if people are irrational then their level of irrationality will be bounded away from zero. However, it is still interesting to analyse what happens in the limit. I would argue likewise with coalitional behaviour, which is to individualism what irrationality is to rationality.

2.3.2 Why model coalitional behaviour as rare?

It is one thing to analyse the behaviour of a dynamic where coalitional behaviour is possible. It is another thing to analyse the predictions of a model where coalitional behaviour becomes infinitesimally likely in the limit. Aside from the pure academic interest as described above here are two reasons why we should be interested. Firstly, people deal with a lot of games in their everyday lives and the amount of time they devote to any given one is by necessity limited. It is not unreasonable to think that it is quite a rare event that two or more people get together and discuss any particular aspect of their lives and credibly agree to make the necessary changes in strategy to better their outcomes. It is also not unreasonable to think that the more people required to agree to changes in behaviour, the harder these changes are to accomplish. Secondly, for our results of this paper it is not necessary that individual strategic switching occur with positive probability in the limit, merely that it is much more likely than coalitional strategic switching. Thus, any type of strategic change can be viewed as a rare event, an assumption which is not unreasonable given the stasis observed in people's behaviour in the overwhelming bulk of the aspects of their lives.

2.3.3 CSS justifies 'experimentation'

As described above, several models have attempted to model random deviations as 'experimentation' by players. The problem here is that such experimentation by a single player is always going to damage the payoff of the experimenting player in the short term. To deal with this, justifications for experimentation have to endow the players with some amount of foresight, for example by suggesting that players will be more likely to experiment when at states which give them relatively poor payoffs because they feel that there must be something better out there. Such justifications sit uneasily will the myopic nature of the rest of the adaptive learning process. CSS however, easily justifies interpretations of deviations as experimentation, as experimenting players can achieve higher payoffs by participating in a coalitional deviation. CSS can thus incorporate the idea of experimentation into stochastic stability notions without departing from the myopia of standard adaptive processes.

2.3.4 CSS can significantly lower convergence times

One of the problems with stochastic stability [2] is the incredibly long times it can take to move between stationary states of the unperturbed process. The reason for this is that players typically need to make several mistakes to push the process from one state to another, the probability of each random error occuring is very small, and the probability of several random errors occuring is much smaller still. Moreover, it is frequently the case that several mistakes need to be made by a single player (or class of players) to push the system to another state. this requires a player to make an error which damages his payoff then continue to make that error several times. With CSS however, we

[2] See for example Ellison (1993)

have every reason to believe that a profitable joint deviating strategy will be continued in future periods. If a coalition of players have tried something and it works, then why not try it again? This paper will show how persistence of profitable coalitional deviations can lead to a considerable reduction in time taken to move between different states of the dynamic process, thus dealing with one of the most important challenges which faces stochastic stability notions.

2.3.5 Realistic modeling of social change

The combination of myopic behaviour with the assumption that coalitional behaviour is more likely to take place with smaller coalitions of players than with larger coalitions can be used to explain aspects of social change. It gives an explanation, for instance, of why revolutionary social movements (large coalitional deviations) will typically have a short life span before breaking down (small coalitional deviations) into something other than what was originally intended.

3 Basic model

This paper shall follow closely the methods and examples of Young (1993).[3] Take an n-player game with finite strategy sets $X_1, \ldots, X_n$; $X = \prod X_i$; and payoffs given by $u_i : X \to \mathbb{R}$. The action taken by player i at time t is denoted x_i^t. The action profile played at time t is denoted $x_t = (x_1^t, \ldots, x_n^t)$. The *state* of the system is given by the actions played in the last m periods and is denoted $h_t = (x^{t-m+1}, \ldots, x^t)$. The higher the value of m the longer the memory of the players. X^m denotes the set of all possible states. The system is taken to start at an arbitrary state: it is assumed at least t periods have already elapsed since the beginning of time.

We define a Markov process P^0 on X^m as follows:

- From state h_t player i draws a random sample of s actions out of the previous m actions taken by each of the other players. These $n-1$ samples are drawn independently. The sample distribution of j's actions in i's sample is denoted $\hat{p}_{ij}^t$ and we write $\prod_{j \neq i} \hat{p}_{ij}^t = \hat{p}_{-i}^t$
- Player i plays a best response to $\hat{p}_{-i}^t$. If there exist tied best responses they are played with equal probability. The actions all players take define x_{t+1} and thus the next state h_{t+1}.

[3] See also Young (1998a), Young (1998b).

Young defines a perturbed version of the process P^ϵ where with probability $1-\epsilon$ a player plays a best response as per P^0. With probability ϵ he instead makes an 'error' and plays a random action from a distribution with full support over his possible actions.

4 Motivating example

In his paper dealing with conventional contracts Young (1998a) gives the example of The Marriage Game.[4] In this game a woman and a man have the option of taking control (TC), sharing control (SC) or ceding control (CC).

		Men		
		TC	SC	CC
	TC	$0,0$	$0,0$	$5,1$
Women	SC	$0,0$	$3,3$	$0,0$
	CC	$1,5$	$0,0$	$0,0$

Young (1998b) shows that the stochastically stable outcome of this game when the sample size is sufficiently large is for men and women to share control. The reason for this is that "conventions with extreme payoff implications are relatively easy to dislodge" because one of the players is "dissatisfied compared to what they could get under some other arrangement". It does not take many stochastic shocks to create an environment in which they prefer to try something different.

Here I expand the marriage game to include more than 2 players. Specifically I look at a version of the game where there is one man and n women playing the game. The man receives his coordination payoff as long as at least one woman coordinates with him. If more than one woman correctly coordinates with the man, each woman to do so "marries" (i.e. receives her coordination payoffs) with equal probability. Players have the same actions available as in the game above. Payoffs are specifically:

$$\pi_M(TC) = \begin{cases} 5, & \text{if at least one woman plays CC} \\ 0, & \text{otherwise} \end{cases}$$

$$\pi_M(SC) = \begin{cases} 3, & \text{if at least one woman plays SC} \\ 0, & \text{otherwise} \end{cases}$$

[4]Note that this is not a particularly esoteric example and can be used to model a number of situations where two agents have to agree on a contract. Instead of a man and a woman in the example we could have a real estate company and a building contractor, with payoffs representing the share of each party's surplus generated by the agreed contract.

$$\pi_M(CC) = \begin{cases} 1, & \text{if at least one woman plays TC} \\ 0, & \text{otherwise} \end{cases}$$

In the following expressions d is the number of women who play the same action as the woman whose payoff is in question (inclusive).

$$\pi_W(TC) = \begin{cases} 5/d, & \text{if the man plays CC} \\ 0, & \text{otherwise} \end{cases}$$

$$\pi_W(SC) = \begin{cases} 3/d, & \text{if the man plays SC} \\ 0, & \text{otherwise} \end{cases}$$

$$\pi_W(CC) = \begin{cases} 1/d, & \text{if the man plays TC} \\ 0, & \text{otherwise} \end{cases}$$

In Young (1998b) stochastically stable states are found by computing the resistances between recurrence classes. These are defined as the smallest number of stochastic shocks required to move from one recurrence class to another. Figure 1 shows what Young calls the *reduced resistances* of the 2-player marriage game. These are the resistances divided by the sample size s.

Young also defines a concept called *stochastic potential*. If each recurrence class of the process is drawn as a vertex on a graph, then for a given recurrence class i we can draw a spanning tree such that from every other recurrence class $j \neq i$ there is a unique path from j to i. The resistances of the edges of this graph can be summed. The stochastic potential of recurrence class i is then defined as the minimum of these sums across all possible spanning trees. The stochastically stable recurrence classes are then shown in Theorem 2 of Young (1993) to be the classes with the lowest stochastic potential.

We wish to calculate the resistances in the n-woman marriage game. There are 3 absorbing states in the n-woman marriage game. I shall call these ♂, ⚥ and ♀ representing male, shared and female control respectively (eg. In ♂ the man plays TC and all the women play CC). There are basically two types of possible transition here: those with M's payoff increasing and those with M's payoff decreasing. Here we examine one instance of each (from which follows the general result).

4.1 M's payoff increasing

Transition: ♀ → ♂

Firstly we ask how many times women must mutate for this transition to take place. Say one woman out of the n women undergoes mutations. We call r the number of mutations necessary for the transition to take place

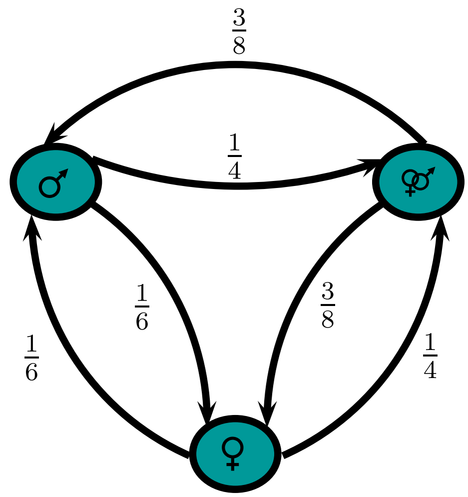

Figure 1: Reduced resistances for 2 player marriage game

with no further mutations. This is the number of mutations required for M to think it worthwhile to play TC after sampling the actions of the women. We examine M's expected payoffs when he samples r plays of CC from the woman in question (the other women do not mutate and continue to play TC.

$$\mathbf{E}[\pi_M(CC)] = 1$$

$$\mathbf{E}[\pi_M(TC)] = 5\frac{r}{s}$$

Hence M might switch from CC to TC when

$$r \geq s\frac{1}{5}$$

In this way we find candidates for reduced resistances:

Transition	r_r ?
♀→ ♂	$\frac{1}{5}$
♀→ ⚥	$\frac{1}{3}$
⚥→ ♂	$\frac{3}{5}$

But now we have to ask if such a transition could instead be caused by rational behaviour by the women. This is never the case in the two player marriage game for transitions that improve M's payoff, but can be the case for the n-woman game as there is a first mover advantage in that if a woman guesses correctly the action of the man and the other women fail to do so a higher relative payoff ensues as it is not shared with the other women.

So say the man has suffered r mutations from CC to TC. Then if the women sample all of these mutations then:

$$\mathbf{E}[\pi_W(TC)] = \frac{5(s-r)}{ns}$$
$$\mathbf{E}[\pi_W(CC)] = \frac{1r}{s}$$

Hence W might switch from TC to CC when

$$r \geq s\frac{5}{n+5}$$

The transition of M to TC can follow without any further mutations.

In this way we find further candidates for reduced resistances:

Transition	r_r ?
♀→ ♂	$\frac{5}{n+5}$
♀→ ⚥	$\frac{5}{3n+5}$
⚥→ ♂	$\frac{3}{n+3}$

We conclude that reduced resistances for the following transitions are:

Transition	r_r
♀→ ♂	$\min(\frac{1}{5}, \frac{5}{n+5})$
♀→ ⚥	$\min(\frac{1}{3}, \frac{5}{3n+5})$
⚥→ ♂	$\min(\frac{3}{5}, \frac{3}{n+3})$

4.2 M's payoff decreasing

Here we need only examine transitions of the type where M mutates and W react rationally to this mutation. Following an identical procedure to the second part of the subsection immediately preceding we obtain the following reduced resistances:

Transition	r_r
♂→ ♀	$\frac{1}{5n+1}$
♂→ ⚥	$\frac{1}{3n+1}$
⚥→ ♀	$\frac{3}{5n+3}$

So we have obtained resistances for every possible transition in our n-woman marriage game.

It is apparent from the diagram and the argument above that there are two effects which make the n-woman marriage game different to the 2-player

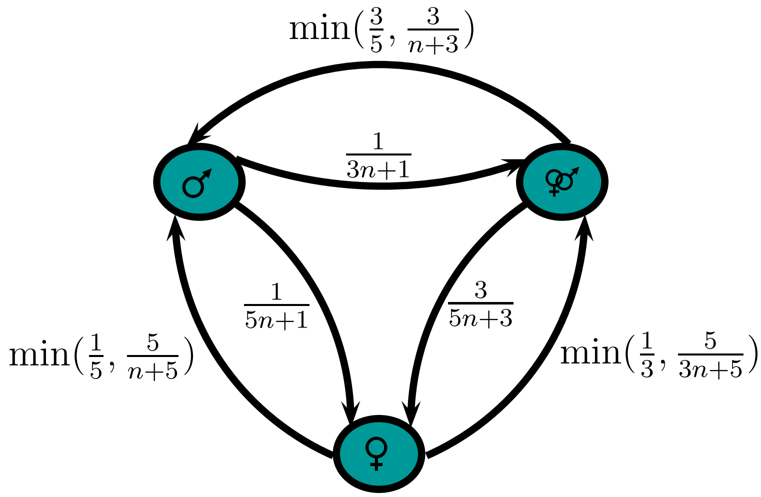

Figure 2: Reduced resistances for n-woman marriage game

marriage game. The first effect is that even when one woman has mutated away from the current convention, there are still sometimes women who remain playing that convention. This increases the expected payoffs to M from sticking with the existing convention and makes the resistances of transitions which are payoff improving for M higher than in the 2-player game. The second effect is that lower payoffs for the women in the n-woman marriage game lead to a greater willingness of individual women to experiment with different actions when they observe behaviour by M that is not in keeping with the current convention. For big enough n this effect will come to dominate the first effect for transitions which improve M's payoffs. However, the second effect is greater for transitions which decrease M's payoffs. The result is that as n increases we would expect to see stochastic stability select conventions which give M lower payoffs. This is counterintuitive: it would seem to be a reasonable real world assumption that increased numbers of women in the marriage market would lead to greater market power for M and allow him to extract higher payoffs.

5 Aye, there's the rub

Results like this continue to occur even when the number of contracts available to the man and the women increase markedly (eg. to sharing 100 units in integer increments). I assert that one of the reasons these strange results occur is because of the probabilities given to given mutation patterns in each period. Within the framework of Young (1993) payoff improving deviations by a single player have a high probability of being realized. Payoff decreasing deviations by a single player have a probability of order ϵ of being realized. Deviations by two players such that either deviation on its own would be detrimental to the deviator's payoff have a probability of order ϵ^2 of being realized, *whether or not the deviations taken together offer a pareto improvement to the deviators*. Thus as $\epsilon \to 0$, joint deviations by two players which offer payoff improvements become infinitely unlikely compared to unprofitable deviations by a single player. This is clearly not satisfactory for all situations, especially situations where coalitional behaviour by groups of players is likely.

Consider the 3-woman marriage game if the situation were reversed, with profitable two player joint mutations occuring with probability of order ϵ and unprofitable single player mutations occuring with probability of order ϵ^2. This effectively doubles the resistances for all the transitions where the payoff of the man decreases and the unique stochastically stable equilibrium in this case as $\epsilon \to 0$ is then ⚥. If we make random errors even more unlikely than joint mutations by increasing the power of epsilon we eventually get ♂ as the unique stochastically stable state. That is, where rational coalitional behaviour is given sufficient priority over irrational single player deviations, the market power given to the man by there being more than one woman in the market is sufficient to take him to his favoured equilibrium (nb. In the above game this happens by a kind of Bertrand argument whereby if the man is not at his optimal contract the man and a woman deviate to a contract where they both do better in the short run and the other women follow so as to earn non-zero payoffs). It is also possible in some games that equilibrium selection can be altered by giving both of the above types of deviation probabilities of the same order (although it doesn't affect the equilibrium chosen in the game above). It should also be noted that in the formulation of Young (1998a) it is always the players who stand to lose from a change of contract who induce the change through their making random errors. Naidu & Bowles (2005) address this by modelling 'experimentation' as a restriction which only allows changes in conventional contracts to be induced by random errors on the part of those who stand to gain from the change. We achieve the same without altering the myopic nature of the model by allowing both parties to gain in the short term from a joint deviation.

6 Coalitional stochastic stability

In this section we introduce our concept of *coalitional stochastic stability*. Like standard stochastic stability it is a concept based on limits and as such shares the benefits of its sharp predictive precision. It also shares the drawback of relying on the the fact that the perturbations in question, - irrationality for SS, coalitional behaviour for CSS - disappear in the limit. However there are some important properties of CSS that SS does not share:

- CSS does not rely on payoff destructive behaviour by agents to gain sharp predictions.
- Versions of CSS can lead to much faster switching between absorbing states of the underlying unperturbed dynamic (discussed in detail later on in the paper).

It should be noted that the version of CSS below *does* include random errors as a technical tool to ensure irreducibility of the Markov process. There are many games where this will make no difference to predictions, and where it does, CSS without random errors will give sharp predictions for large classes of states - precisely those classes of states which are closed under rational coalitional behaviour.

First define the following notation.

$\mathcal{P}$ is the set of all subsets of N.

$\mathcal{P}_m \subset \mathcal{P}$ is the set of all subsets P_m of N such that $|P_m| = m$.

Let F_m be a probability distribution over $\mathcal{P}_m$ with full support.

Given an action profile a and a set $Q \subset N$, let

$$A_Q(a) = \{x : x_i = a_i \ \forall i \notin Q, \pi_i(x) \geq \pi_i(a) \ \forall i \in Q\}.$$

Let $G(A_Q(a))$ define a probability distribution over $A_Q(a)$ with full support.

Let $\hat{H}$ be a probability distribution with full support over N and let H_i be distributions with full support over all possible actions of player i.

Consider the following perturbed adaptive process P^ϵ with $\epsilon_i > 0 \ \forall i$

- With probability $1 - \sum_{i=2}^{n+1} \epsilon_i$ players follow the adaptive learning process as usual.

- With probability ϵ_m there is a Pareto superior deviation by $1 < m \leq n$ players. To be precise, a set of players P_m is selected according to F_m. Actions remain fixed at those of the last period for all $i \in N \backslash P_m$. The actions of $i \in P_m$ change so that payoffs for these players are weakly better under the new action profile than under the old one: given that an action profile x_{t-1} was played in the previous period, a new action profile x_t is selected from the distribution $G(A_{P_m}(x_{t-1}))$ and played in the current period.

- With probability ϵ_{n+1} a random error occurs to the strategy of a randomly selected player. A player is selected according to $\hat{H}$ and he plays an action determined by H_i. The actions of all other players remain the same as in the previous period.

As the process is irreducible it has a unique stationary distribution which I denote (à la Young (1998b)) as μ^ϵ. A state z is *coalitionally stochastically stable* if:

$$\lim_{\epsilon_2 \to 0} \lim_{\epsilon_3 \to 0} \lim_{\epsilon_4 \to 0} \dots \lim_{\epsilon_n \to 0} \lim_{\epsilon_{n+1} \to 0} \mu^\epsilon(z) > 0.$$

6.1 What does this mean?

Effectively I have ranked the different types of deviations in order of importance. Most important are profitable single player deviations, followed by profitable two player deviations and so on. Least important of all are random unprofitable deviations. This order of importance is given by the order in which limits are taken. In working out our CSS states we take in turn:

- Any type of rational coalitional deviation to be infinitely more likely than unprofitable deviations.

- Rational coalitional deviations involving fewer players to be infinitely more likely than rational coalitional deviations involving more players.

So it is apparent that this concept puts a high value on rationality relative to the approach of Young (1993). Conversely it puts lower emphasis on the independence of the players' actions.

7 Propositions

Proposition 1. [5] *CSS states exist and are identical to those selected by the following process:*

[5] See appendix for proof of propositions 1 and 2.

- *Take the recurrence classes defined by the rational coalitional deviations. Find the recurrence class(es) with the lowest stochastic potential with respect to random unprofitable deviations (i.e. those with probability of order ϵ_{n+1}.*

- *Within this recurrence class(es) take the recurrence classes defined by the rational coalitional deviations of order $n-1$ or less. Find which of these have the lowest stochastic potential with respect to coalitional deviations of order n (i.e. those with probability of order ϵ_n. Select these recurrence classes.*

- *Repeat with classes of order $n-2$ and deviations of order $n-1$*

- *Keep going until you have selected recurrence classes of order 1.*

- *These are the coalitionally stochastic states of the underlying game. They comprise a subset of the recurrence classes of the process P^0 and, if singletons, are Nash Equilibria of the underlying game.*

Proposition 2. *Under conditions which in Young (1993) guarantee the selection of a convention(s) by SS, CSS will also select a convention(s). These conditions are:*

- *The underlying game Γ is weakly acyclic.*

- *The sample size s is sufficiently small relative to m.*

8 Example

	L	M	N	O	R
a	$6,2$	$4,1$	$30,0$	$0,\epsilon$	$0,\epsilon$
b	$1,8$	$3,7$	$0,0$	$0,\epsilon$	$0,\epsilon$
c	$0,\epsilon$	$0,\epsilon$	$0,0$	$7,3$	$1,5$
d	$0,\epsilon$	$0,\epsilon$	$0,0$	$8,1$	$2,6$

Figure 3: A two player strategic game.

Here we demonstrate how the algorithm in Proposition 2 works. ϵ is assumed to be very close to zero and is included so that action N is strictly dominated for player 2. The 4x5 game in Figure 3 has 2 strict Nash equilibria, aL and dR, each of which corresponds to a convention of our unperturbed dynamic. Under the standard random errors approach, convention aL will then be selected as it takes relatively few errors by player 2 where he chooses N (a strictly dominated action for him) before it becomes worthwile for player 1 to play a in the hope of earning a very big payoff at aN. Once a is being played by player 1 it then becomes a best response for player two to play L and the convention aL is reached. Under CSS it is clear that there is only one recurrence class under coalitional deviations of 2 players or fewer (i.e. those occuring with probabilities of order $\geq \epsilon_2$) and that this recurrence class includes both conventions: from convention dR enough two player deviations to bM will allow aL to be reached, and from convention aL enough two player deviations to cO will allow dR to be reached. As there exists only one recurrence class under coalitional deviations of 2 players or fewer there is no need to use random errors to choose between such recurrence classes. Within this recurrence class there are two recurrence classes under coalitional deviations of 1 player or fewer (i.e. those occuring with probabilities of order 1) and these recurrence classes are the conventions aL and dR. We now use deviations of order 2 to choose between these conventions. From dR the only possible two player coalitional deviation is to bM and $\frac{s}{3}$ of these are required before it is possible that player 1 sampling from player 2's actions sees player two playing M often enough for player 1 to judge it worth his while to play a. If player 1 the plays a for a while, player 2 can begin to play L and aL is reached. On the other hand, to move from aL to dR via plays of cO requires only $\frac{2s}{7}$ joint deviations before player 2 can switch to playing R followed by player 1 switching to d. So our algorithm selects dR as the unique coalitionally stochastically stable convention.

We have chosen between two conventions without either player engaging in behaviour that is detrimental to his short term myopic best interest. Irrational behaviour is not always necessary in order for stochastic stability arguments to have bite.

9 2×2 Games

Two player games are a special case when it comes to examining coalitional behaviour because any coalitional move in a 2 player game is by definition a move towards efficiency. In fact, where Γ is a 2 player normal form game with multiple strict Nash Equilibria CSS will never select an equilibrium which is not Pareto efficient. If the game also satisfies the conditions of Proposition 2 then CSS effectively eliminates the possibility of the selection of inefficient equilibria and allows a standard stochastic stability notion to choose between the remaining (efficient) equilibria. This leads to the following result.

Proposition 3. *When Γ is a 2×2 game with more than a single strict Nash Equilibrium, CSS induces the following lexicographic decision rule:*

(i) Where one equilibrium is Pareto superior to the other, the superior equilibrium is selected.

(ii) Where neither equilibrium is Pareto superior to the other, the risk dominant equilibrium is selected.

Proof. A 2×2 game with two strict NE is clearly weakly acyclic, so Proposition 2 is satisfied and a convention will be chosen. If one equilibrium is Pareto superior to the other it is the unique recurrence class of $P^{\epsilon_{\leq 2}}$ and so is selected by CSS. If neither NE is Pareto superior to the other both of them are singleton recurrence classes in $P^{\epsilon_{\leq 2}}$. Clearly whichever of them is selected by the random errors of order ϵ_3 will be the selected convention. Effectively we are selecting our CSS states only using the random errors and the selection is reduced to the standard stochastic stability notion which Young (1993) shows selects the risk dominant equilibrium in 2×2 games. □

It should be noted that although the elimination of inefficient equilibria extends to 2-player games with more than 2 strategies per player, the risk dominance part of the result does not. I also note that this efficiency guarantees that the selected NE is coalition proof and part of the core of Γ. The question naturally arises as to how far we can extend our results regarding this preference of CSS for efficiency.

10 Efficiency with >2 players

In the example of the n-woman marriage game the selected equilibrium payoff vector is $\left(5,\ \frac{1}{n},\ \ldots,\ \frac{1}{n}\right)$. This is an element of the core of the game, defined as in Aumann & Peleg (1960), whether the core for games with non-transferable utility is described using the concept of α-efficiency or β-efficiency.[6] It is a well known property of the core that its elements are efficient outcomes of the underlying game. Can we establish any kind of inclusion relation between CSS and the core? The answer is no, we cannot. Even when CSS selects a unique equilibrium we cannot guarantee that this equilibrium is contained in the core of the game. This result is in contrast to Konishi & Ray (2003)

[6] α-efficiency guarantees coalitions payoffs at least as high as their maximin payoffs, β-efficiency guarantees coalitions payoffs at least as high as their minimax payoffs. The core under α-efficiency is the set $\{\left(5,\ \frac{1}{n},\ \ldots,\ \frac{1}{n}\right),\left(3,\ \frac{3}{n},\ \ldots,\ \frac{3}{n}\right),\left(1,\ \frac{5}{n},\ \ldots,\ \frac{5}{n}\right)\}$; the core under β-efficiency is $\{\left(5,\ \frac{1}{n},\ \ldots,\ \frac{1}{n}\right)\}$.}

where a farsighted dynamic process always selects payoffs in the core of the game when a unique limit of the process exists.[7] Serrano & Volij (2005) demonstrate that stochastic stability does not necessarily select equilibria in the core.[8] Here I give an example of a game with a nonempty singleton core and a singleton CSS set which are not the same.

A

	L	R
a	$4, 4, 4$	$2, 0, 6$
b	$0, 6, 2$	$5, 5, 0$

B

	L	R
a	$6, 2, 0$	$0, 1, 5$
b	$5, 0, 1$	$3, 3, 3$

Figure 4: A three player strategic game, in which player 3 chooses A or B.

In the game in Figure 4 the the unique element of the core is aLA with payoffs $\left(4,\ 4,\ 4\right)$. CSS chooses bRB - the only inefficient pure strategy combination possible! The reason for this is that it requires a 3-player coalition to move from bRB to aLA, whereas a 2-player coalition will deviate from aLA to bRA from where a best response of player 3 is to move to bRB (all usual provisos about sample sizes in the adaptive process apply).

10.1 Coalition proofness

Nor is there an inclusion relation between CSS outcomes and Coalition Proof outcomes Bernheim *et al.* (1987). The game in Figure 4 has bRB as its unique CSS outcome and aLA as its unique Coalition Proof outcome.

aLA is coalition proof as all coalitional deviations lead to further deviations by subsets of the deviating players.[9] bRB is not coalition proof as the players can jointly deviate to aLA which is itself coalition proof. However, although a deviation from the Coalition Proof equilibrium aLA to bRA is disturbed by further deviations, it still allows the possibility of a transition to bRB by single player best responses. Thus, coalitions that are not viable deviations in the Coalition Proof equilibrium concept can change outcomes in the CSS concept if they open up opportunities to enter the basin of attraction of another equilibrium.

[7] However, the restrictions Konishi & Ray (2003) place on the dynamic processes in their paper are extremely restrictive and incorporate characteristic functions directly into their definitions.

[8] This is clearly true for games with non-transferable utility. There is however more reason to suspect that CSS and the core might be related, both being defined using coalitional concepts.

[9] Deviation to bRA leads to deviation to bLA.

11 Structure of coalitional deviations

In my definition of CSS I assume that all possible coalitions have the chance to deviate. This can easily be altered to model situations where certain players are not expected to cooperate with one another. An example of this might be a game with a set of buyers and a set of sellers, where sets of sellers can make coalitional deviations (modelling collusive behaviour) but no other set of players can. We can in fact define any hierarchy of subsets of players ordered by the likelihood of the occurence of coalitional behaviour in them and thus by the order in which limits will be taken to select CSS states:

$$\xi_1, \xi_2....., \xi_M, \quad M \in \mathbb{N},\ \xi_i \subset \mathcal{P}\ \forall i$$

Naturally, some coalition structures can be considered more reasonable than others. Suggestions have been made in the cooperative game theory literature that subsets of coalitions which are allowed to deviate should also be allowed to deviate[10] [11] or alternatively that the union of coalitions which are allowed to deviate and have a nonempty intersection should also be allowed to deviate[12]. In the first case it is argued that it is possible for subset of a set of players who meet to discuss strategy to meet without the others present. The second case is predicated on the argument that players who are members of the intersection between two coalitions can serve as intermediaries to bring the interests of the two coalitions together.

12 Further examples

12.1 Bertrand game with discrete pricing

It is well known that a Bertrand game with two firms each with unit cost of production c who can set prices in integer increments has multiple Nash equilibria. Assuming that if each firm sets the same price they share the market equally then there is an equilibrium where each firm charges c, an equilibrium where each firm charges $c+1$, and an equilibrium where each firm charges $c+2$. We assume a dynamic as per our model with $s = m = 1$. From any of the equilibria it is of benefit to both firms to jointly deviate

[10] Of course the argument is not phrased in this manner in cooperative papers. Instead the structure of allowable deviations in cooperative games is described by the set of characteristic function inequalities that need to be satisfied in order for a game to be counted as part of the core of the game or to satisfy another cooperative solution concept such as the nucleolus (Schmeidler (1969))

[11] Algaba *et al.* (2000)

[12] Algaba *et al.* (1999)

to charging some higher price $c + x$ with $x > 2$. However, then under our dynamic (which with $s = m = 1$ is a simple best response dynamic) the firms then attempt to undercut one another until the equilibrium where they both charge $c + 2$ is attained. However at this equilibrium the best responses are not unique and charging $c+1$ when the other firm is charging $c+2$ is also a best response. So the dynamic will also eventually take us to the equilibrium where both firms charge $c+1$. Notice that as long as both firms are charging at least $c + 1$ then charging a price of c or lower is never a best response. This means that the equilibrium where both firms play c is eliminated from the recurrence class of states under deviations of two players or fewer. Then letting the probability of two player deviations become infinitesimally small we see that the probability of leaving the equilibrium where both firms charge $c + 1$ goes to zero, whereas to get to this state from the other states does not require joint deviations. Hence both firms charging $c + 1$ is the unique coalitionally stochastically stable state.

13 More on 2×2 games

To use the ideas behind CSS, it is not even necessary to use the lexicographic tool of ordered limits. Here I examine the special case of 2×2 games when the probability of coalitional deviation and the probability of random error deviation are closely related. That is, we take there to be an ϵ probability of a random error occuring and a ϵ^λ probability of a 2-player coalitional deviation occuring, where $\lambda > 0$. This examination is only going to be interesting when there is some tension between efficiency and risk dominance so we examine games with one efficient strict NE and one risk dominant NE.

		Player 2: 1	Player 2: 2
Player 1	1	a_{11}, b_{11}	a_{12}, b_{12}
	2	a_{21}, b_{21}	a_{22}, b_{22}

We assume without loss of generality that (1,1) and (2,2) are strict NE and that (1,1) is Pareto superior:

$$a_{11} > a_{22}\,, \qquad b_{11} \geq b_{22}$$

We also assume that (2,2) is the risk dominant equilibrium:

$$(a_{22} - a_{12})(b_{22} - b_{12}) > (a_{11} - a_{21})(b_{11} - b_{21})$$

Proposition 4. *In a 2×2 game with 2 strict NE, one efficient and one risk dominant, the efficient equilibrium is selected if and only if:*

$$\lambda \leq \min\{\frac{\beta}{1-\alpha}, \frac{\alpha}{1-\beta}\}$$

with

$$\alpha = \frac{(a_{11} - a_{21})}{(a_{11} - a_{12} - a_{21} + a_{22})}, \qquad \beta = \frac{(b_{11} - b_{12})}{(b_{11} - b_{12} - b_{21} + b_{22})}$$

Proof. Resistances measured in terms of necessary deviations of probability ϵ are:

$$r_{12}^s = \min\{\lceil \alpha s \rceil, \lceil \beta s \rceil\}$$

and[13]

$$r_{21}^s = \min\{\lceil (1-\alpha)s \rceil, \lceil (1-\beta)s \rceil, \lambda\lceil (1-\alpha)s \rceil, \lambda\lceil (1-\beta)s \rceil\}$$

Equilibrium $(1,1)$ is selected when $r_{12}^s > r_{21}^s$. For s large enough (so that we can ignore the $\lceil \cdot \rceil$ operators), this gives us the condition stated in the proposition.

□

14 Experimentation

It is mentioned above that CSS gives added weight to the explanation of the small probability events underlying SS as experimentation. When there is a tiny chance (the chance that another player simultaneously randomly experiments) of 'experimentation' leading to increased payoffs it seems a funny kind of behaviour for players to engage in. For this reason, justifications usually involve arguments that depart from the myopia of standard adaptive learning. However, with coalitional behaviour it becomes possible for there to be a non-negligible possibility of 'experimentation' resulting in increased payoffs for the players concerned. For completeness we outline a slightly different model to our one above which better fits this interpretation.

Consider the following perturbed adaptive process E^ϵ with $\epsilon_i > 0 \ \forall i$

- There is an action profile being played at the start of the period.

[13] This is simply the equivalent expression without coalitional deviations $\min\{\lceil (1-\alpha)s \rceil, \lceil (1-\beta)s \rceil$ with adjustments made for the case when it is easier to shift via coalitional deviations to $(1,1)$ rather than via error deviations.

- With probability $1 - \sum_{i=1}^{n+1} \epsilon_i$ nothing happens.
- With probability ϵ_m m players are selected randomly and choose random actions. They keep these actions if they all do at least as well under the new action profile as under the existing action profile. Otherwise they revert to the actions they were playing at the start of the period.

So we have a model of experimentation to which stochastic stability arguments can be applied which does not rely on irrational short term behaviour by players.

15 Time to convergence

To some degree the arguments in this section are trivial. However, this does not stop them from being of some importance. One of the greatest problems affecting stochastic stability arguments is the large amounts of time it can take on average to move to a stochastically stable convention from a non-SS convention. Consider the following game:

		Player 2: 1	Player 2: 2
Player 1	1	$10, 10$	$0, 0$
	2	$0, 0$	$8, 8$

If $s = 9$ then four mutations by either player are required to move from the convention $(2, 2)$ to the convention $(1, 1)$. If $\epsilon = \frac{1}{10}$ we can expect this to happen over any given four periods with probability of order $\epsilon^4 = \frac{1}{10000}$. That is, a lot of time is expected to elapse before a move to $(1, 1)$. With CSS however, it is not too hard to imagine that following a coalitional deviation to $(1, 1)$ further coalitional deviations might be more likely or even certain. This would trivially cut the expected time before a move to $(1, 1)$ occurs. Such an assumption makes sense because even if the 'players' are emerging from populations of agents, you would expect such successful behaviour to have some chance of being communicated and thus replicated in the following period. This argument cannot be used with standard SS: in fact under SS you might expect errors to become even more unlikely following an error given the damaging effect of errors on immediate payoffs.

We stress that this argument is not opposed to those of Young (1993) but rather complementary to them. If we understand the model as being that of the same game being played repeatedly between representatives of different populations then the assumptions of uniform errors and no coalitional behaviour become more realistic the larger the underlying populations. The arguments giving CSS faster convergence times are stronger the smaller the underlying population and the better the communication of successful strategies within such populations. This brings us to the next section, which takes a look at an area where standard stochastic stability is (we argue) an inappropriate but frequently used tool.

16 Local interaction

> *"Local matching rules are appropriate to describe situations where players interact not with the population as a whole, but rather with a few close friends or colleagues. For example, such a rule might describe the interactions at a college reunion where each participant knows in advance who he or she wishes to see." Ellison (1993)*

Stochastic stability has been used in a variety of local interaction models, for example Ellison (1993),Ellison (2000), Goyal & Vega-Redondo (1999),Bala & Goyal (2000). Ellison (1993) notes that convergence to stochastically stable states can be much faster when players interact with a small group of neighbours than when they are uniformly matched across an entire population. Ellison notes that it is possible for models to have convergence in reasonable time and gives an example where this is indeed the case: local interaction models of coordination where each player can choose between two strategies A and B and wishes to play strategy A if and only if at least a certain proportion of his neighbours are also playing strategy A.

As an immediate caveat to the above observation we would like to note that time to convergence in uniform error models can depend enormously on the number of strategies available to the players concerned. This is easy to miss as it is irrelevant in the limit as the probability of an error occuring approaches zero. As an example consider 5 players who have also committed errors (for a given set of 5 players this occurs with probability ϵ^5 in a uniform error model). Let us say that when acting in error a player has no preference over the action he plays. Then if players have 2 actions available to them, they will all play the same action with probability $(\frac{1}{2})^4 = \frac{1}{16}$. However, if they have 10 actions available they will all play the same action with probability $(\frac{1}{10})^4 = \frac{1}{10000}$. It is clear from this that the number of actions available can have a large effect on convergence times as helpful coordination of errors becomes less likely. It can be immediately seen that this problem does not arise in CSS, where although coalitional behaviour may be a small probability event, when it does take place coordination will be automatic [14]. Applying the structure of local interaction models to coalitional behaviour is also easy: we simply take coalitional behaviour by players who are not 'close' to one another to be impossible. Table 1 outlines the reasons why we find coalitional arguments superior to standard SS in local interaction models.

SS	**CSS**
Movement between conventions depends on a degree of unpredictability in the actions of a player's neighbours, who he supposedly knows reasonably well.	Actions leading to movement between conventions are determined by rational behaviour jointly agreed by players who know one another.
Spread of new ideas and conventions is an essentially random process.	Network effects and the spread of ideas are modelled in a rational way.
New technology modelled as being *randomly* adopted by multiple agents at once.	Can model decisions to introduce new technologies as being jointly made by those who stand to benefit.

Table 1: Local interaction: SS vs. CSS

17 Conclusion

We have presented the use of coalitional stochastic stability as a method of equilibrium selection, and argued that it should be preferred to random error based stochastic stability wherever coalitional behaviour is feasible. We have demonstrated that the ideas underlying CSS are as intuitive as those underlying standard stochastic stability and shown how CSS states can be found. CSS is a way of incorporating coalitional considerations into equilibria and thus falls into the same strand of literature as papers on coalition proofness. We have shown how despite a strong preference for efficiency in the description of CSS, efficiency will not always be attained and that sometimes social

[14] Although of course this is not necessarily the case in models such as our 'experimentation' model given above

movements to other Nash equilibria (such as the French Revolution or the move from bRB to aLA in Figure 4) will quickly collapse due to further deviations. This does not mean that such changes will not happen. They will happen, and the resulting instability may take you somewhere new.

We have demonstrated that unlike standard stochastic stability approaches, a CSS approach does not rely on payoff destructive behaviour by individual players, and this helps us to justify the interpretation of random behaviour as experimentation. The payoff beneficial nature of CSS deviations also allows models to be built which converge much faster to the stable states than has been the case with previous versions of stochastic stability, reducing history dependence and enabling the modelling of social change on a more realistic timescale.

Coalitional behaviour is something that can be observed in many noncooperative games and in discussions of Nash and other equilibrium concepts it is often the elephant in the room: the question being how to incorporate the realism of coalitional behaviour without discarding the precision of equilibrium predictions. This paper has given one way of overcoming this problem. It would be interesting to see further analysis of games with coalitions and a multiplicity of equilibria using the tools described in this paper.

References

Algaba, E., Bilbao, J.M., & Borm, P. 1999. *The Myerson value for union stable systems*. Tech. rept.

Algaba, E., Bilbao, J.M., & Brink, R. van den. 2000. *Cooperative games on antimatroids*. Tech. rept.

Aumann, R. 1959. *??Acceptable Points in General Cooperative n-Person Games,?? in Contributions to the Theory of Games IV, (A. W. Tucker and R. D. Luce, Eds.)*. Tech. rept.

Aumann, Robert, & Peleg, Bezalel. 1960. Von Neumann-Morgenstern Solutions to Cooperative Games without Side Payments. *Bulletin of the American Mathematical Society*, **66**.

Bala, Venkatesh, & Goyal, Sanjeev. 2000. A Noncooperative Model of Network Formation. *Econometrica*, **68**(5), 1181–1230.

Bergin, James, & Lipman, Barton L. 1996. Evolution with State-Dependent Mutations. *Econometrica*, **64**(4), 943–56.

Bernheim, B. Douglas, Peleg, Bezalel, & Whinston, Michael D. 1987. Coalition-Proof Nash Equilibria I. Concepts. *Journal of Economic Theory*, **42**(1), 1–12.

Ellison, Glenn. 1993. Learning, Local Interaction, and Coordination. *Econometrica*, **61**(5), 1047–71.

Ellison, Glenn. 2000. Basins of Attraction, Long-Run Stochastic Stability, and the Speed of Step-by-Step Evolution. *Review of Economic Studies*, **67**(1), 17–45.

Goyal, S., & Vega-Redondo, F. 1999. *Learning, network formation and coordination*. Tech. rept.

Kandori, Michihiro, Mailath, George J, & Rob, Rafael. 1993. Learning, Mutation, and Long Run Equilibria in Games. *Econometrica*, **61**(1), 29–56.

Konishi, Hideo, & Ray, Debraj. 2003. Coalition formation as a dynamic process. *Journal of Economic Theory*, **110**(1), 1–41.

Naidu, Suresh, & Bowles, Samuel. 2005. Equilibrium Selection by Intentional Idiosyncratic Play. *Santa Fe Institute working paper*, August.

Peleg, B., & Sudholter, P. 2003. Introduction to the Theory of Cooperative Games, ISBN 1-4020-7410-7 378 pages. *Kluwer Academic, Boston*.

Schmeidler, David. 1969. The nucleolus of a characteristic function game. *SIAM Journal of Applied Mathematics*, **17**.

Serrano, Roberto, & Volij, Oscar. 2005 (Nov.). *Mistakes In Cooperation: The Stochastic Stability Of Edgeworth'S Recontracting*. Economics Working Papers we056332. Universidad Carlos III, Departamento de Economa.

van Damme, Eric, & Weibull, Jorgen W. 2002. Evolution in Games with Endogenous Mistake Probabilities. *Journal of Economic Theory*, **106**(2), 296–315.

Young, H Peyton. 1993. The Evolution of Conventions. *Econometrica*, **61**(1), 57–84.

Young, H Peyton. 1998a. Conventional Contracts. *Review of Economic Studies*, **65**(4), 773–92.

Young, H. Peyton. 1998b. Individual strategy and social structure.

18 Appendix

Proof of propositions 1 and 2. Define $P^{\epsilon_{\leq n}}$ as identical to P^{ϵ} in all respects other than that $\epsilon_{n+1} = 0$. Denoting the unique stationary distribution of P^{ϵ} as μ^{ϵ} for each $\epsilon_{n+1} > 0$, we know from Theorem 3.1 of Young (1993) that $\lim_{\epsilon_{n+1}\to 0} \mu^{\epsilon} = \mu^{\epsilon_{\leq n}}$ exists and $\mu^{\epsilon_{\leq n}}$ is a stationary distribution of $P^{\epsilon_{\leq n}}$. We also know that the states z with $\mu^{\epsilon_{\leq n}} > 0$ are contained in the recurrence classes of $P^{\epsilon_{\leq n}}$ with the lowest stochastic potential. Take one of these recurrence classes and call it $\mathcal{Z}$. The process $P^{\epsilon_{\leq n}}$ restricted to $\mathcal{Z}$ is clearly irreducible and positive recurrent. Hence it has a unique stationary distribution which must be $\mu^{\epsilon_{\leq n}}$ restricted to the states in $\mathcal{Z}$ and scaled so as to sum to 1. We denote this distribution $\mu_{\mathcal{Z}}^{\epsilon_{\leq n}}$. Define $P^{\epsilon_{\leq n-1}}$ as identical to $P^{\epsilon_{\leq n}}$ except that $\epsilon_n = 0$. Define $P_{\mathcal{Z}}^{\epsilon_{\leq n}}$ and $P_{\mathcal{Z}}^{\epsilon_{\leq n-1}}$ as these processes restricted to $\mathcal{Z}$. Then reiterating Young's Theorem we have that $\lim_{\epsilon_n\to 0} \mu_{\mathcal{Z}}^{\epsilon_{\leq n}} = \mu_{\mathcal{Z}}^{\epsilon_{\leq n-1}}$ exists and is a stationary distribution of $P_{\mathcal{Z}}^{\epsilon_{\leq n-1}}$. As this applies to every possible recurrence class $\mathcal{Z}$ of $P^{\epsilon_{\leq n}}$ we then have that $\lim_{\epsilon_n\to 0} \mu^{\epsilon_{\leq n}} = \mu^{\epsilon_{\leq n-1}}$ exists and is a stationary distribution of $P^{\epsilon_{\leq n-1}}$. Now if we regard deviations of n-players occuring with probability of order ϵ_n as the deviations used to measure stochastic potential, it is immediately clear that for any $\mathcal{Z}$, the states $z \in \mathcal{Z}$ such that $\mu_{\mathcal{Z}}^{\epsilon_{\leq n-1}} > 0$ are the states with the lowest stochastic potential.

Continuing in this fashion we see that

$$\lim_{\epsilon_2\to 0} \lim_{\epsilon_3\to 0} \lim_{\epsilon_4\to 0} \ldots \lim_{\epsilon_n\to 0} \lim_{\epsilon_{n+1}\to 0} \mu^{\epsilon}(z) = \mu^0(z)$$

exists and is a stationary distribution of P^0. It is clear from the above that the states z with $\mu^0(z) > 0$ can be calculated using the process described in Proposition 1. Given that CSS selects a stationary distribution of P^0 and that we know from Young (1993) that stationary distributions of P^0 select convention(s) under the conditions given in Proposition 2, the proof is complete.

□