Liquidity Shortages and Monetary Policy

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Abstract

The paper models the interaction between risk taking in the financial sector and central bank policy. It shows that in the absence of central bank intervention, the incentive of financial intermediaries to free ride on liquidity in good states may result in excessively low liquidity in bad states. In the prevailing mixed-strategy equilibrium, depositors are worse off than if banks would coordinate on more liquid investment. It is shown that public provision of liquidity improves the allocation, even though it encourages more risk taking (less liquid investment) by private banks.

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1. Introduction

1.1 The Issues

Recently, many market participants have the feeling that financial markets have been susceptible to excessive risk taking, encouraged by extremely low (some even claim artificially depressed) risk spreads. There is the notion of abundant liquidity, stimulated by a "savings glut"; by an "investment restraint" or by central banks running too-loose monetary policies. At the same time, quite a few market participants see the rising risk of a liquidity squeeze which would force central banks to ease policy again (compare, for example, *A fluid concept, The Economist* (2007)). Frequently it is argued that it is exactly the anticipation of such a central bank reaction which encourages further excessive risk taking: The belief in "abundant" provision of aggregate liquidity is supposed to be the source of all evils, resulting in overinvestment in activities creating systemic risk.

This paper models the interaction between risk taking in the financial sector and central bank policy. It shows that in the presence of aggregate risk, banks may take no precaution against the risk of being run in bad states, resulting in inefficient bank runs. In that case, public provision of liquidity can improve upon the allocation, even though it encourages more risk taking (less liquid investment) by private banks. Integrated markets depend on a reliable superstructure for the availability of liquidity under stress. Liquidity provision by central banks provides an insurance against aggregate risk, encouraging investment in risky projects with higher return. Furthermore, we show that the incentive of financial intermediaries to free ride on liquidity in good states may result in excessively low liquidity in bad states. In the prevailing mixed-strategy equilibrium, depositors are worse off than if banks would coordinate on more liquid investment.

Liquidity provision, rather than encouraging excessive risk taking, actually can improve the allocation in an incomplete market economy. Such a policy, however, may impede the role of money as a medium to facilitate ordinary transactions. Thus, the real trade off is between price stability and financial stability: the role of monetary policy as insurance against financial fragility may be in conflict with the traditional task of controlling inflation when the central bank has just one instrument to achieve both goals.

The present paper builds on the set up of **Diamond/ Rajan** (2006) and extends it to capture the feedback from liquidity provision to risk taking incentives of financial intermediaries. As in Diamond/ Rajan, deposit contracts solve a hold up problem for impatient lenders investing in illiquid projects: these contracts give banks as financial intermediaries a credible commitment mechanism not to extract rents from their specific skills. But at the same time deposit contracts make non-strategic default very costly. Consequently, negative aggregate shocks may trigger banks runs with serious costs for the whole economy, thus destroying the commitment mechanism. Diamond /Rajan (2006) show that monetary policy can alleviate this problem in an economy with nominal deposits: Via open market operations, the central bank can mimic state contingent real debt contracts by adjusting the nominal price level to the size of the aggregate shock.

The paper extends the set up of Diamond/Rajan in several ways. In their model, the type of risky projects is exogenously given. Banks can either invest in risky, possibly illiquid projects or invest instead in a safe liquid asset with inferior return. In the equilibrium they characterise, banks invest all resources either in illiquid or liquid assets. They do not analyse the feedback mechanism from monetary policy towards the risk taking of financial intermediaries when central bank policy works as insurance mechanism against aggregate risk.

In contrast, the present paper determines endogenously the aggregate level of illiquidity out of private investments. As in Diamond/ Rajan, illiquidity is captured by the notion that some fraction of projects turns out to be realised late. In contrast to their approach, however, we allow banks to choose continuously the proportion of funds invested in less liquid projects. These projects have a higher expected return, but at the same time also a higher probability of late realisation.

It will be shown that insuring against aggregate risks results in a higher share of less liquid projects funded. So liquidity provision as public insurance does indeed encourage higher risk taking, superficially confirming the notion of “excessive” risk. It turns out, however, that – as long as there is no conflict with money being a stable medium of transaction – this policy is welfare improving rather than creating moral hazard. If, however, liquidity provision has adverse impact on price stability, there may be a trade off for the central bank. The trade off is due to the fact that the central bank has to achieve two goals (price stability and financial stability) with just one instrument (injection of liquidity).

1.2 Related Literature

Liquidity provision has been mainly analysed in the context of models with real assets - see **Diamond/ Dybvig** (1983), **Bhattacharya/ Gale** (1987), **Diamond/ Rajan** (2001, 2005), **Fecht/ Tyrell (2005)** and for a survey the reader of **Goodhart/Illing** (2002). Only a few recent papers explicitly include nominal assets and so are able to address monetary policy (**Allen/ Gale** (1998), **Diamond/ Rajan** (2006), **Skeie** (2006) and **Sauer** (2007). **Skeie** (2006) shows that nominal demand deposits, repayable in money, can prevent self-fulfilling bank runs of the **Diamond/ Dybvig** type, when interbank lending is efficient.

Here, we are concerned with bank runs triggered by real shocks as in **Diamond/ Rajan** (2006). Demand deposits provide a credible commitment mechanism. A related, but quite different mechanism has been analysed by **Holmström/ Tirole** (1998). They model credit lines as a way to mitigate moral hazard problems on the side of firms. In their model, **Holmström/ Tirole** also characterise a role for public provision of liquidity, but again they do not consider feedback mechanisms creating endogenous aggregate risk.

Apart from **Diamond/ Rajan** (2006), the paper most closely related is **Sauer** (2007). Building on the cash-in-the-market pricing model of **Allen /Gale** (2005), Sauer analyses liquidity provision by financial markets and characterises a trade-off between avoiding real losses by injecting liquidity and the resulting risks to price stability in an economy with agents subject to a cash-in-advance constraint. The present paper uses the more traditional framework with banks as financial intermediaries. This framework can capture the impact of financial regulation in a straightforward way.

1.3 Sketch of the Paper

Section 2 presents the basic settings of the model. Let us here sketch the structure already informally. There are three types of agents, and all agents are assumed to be risk neutral.

(1) **Entrepreneurs**. They have no funds, just ideas for productive projects. Each project needs one unit of funding in the initial period 0 and will either give a return early (at date 1) or late (at date 2). There are two types of entrepreneurs: Entrepreneurs of type 1 with projects maturing for sure early at date 1, yielding a return $R_1 > 1$ and entrepreneurs of type 2 with projects yielding a higher return $R_2 > R_1 > 1$. The latter projects, however, may be delayed: With probability $1 - p$, they turn out to be illiquid

and can only be realised at date 2. For projects being completed successfully, the specific skills of the entrepreneur are needed. Human capital being not alienable, entrepreneurs can only commit to pay a fraction $\gamma R_i > 1$ to lenders. They earn a rent $(1-\gamma) R_i$ for their specific skills. Entrepreneurs are indifferent between consuming early or late.

(2) **Investors.** They have funds, but no productive projects on their own. They can either store their funds (with a meagre return 1) or invest in the projects of entrepreneurs. Investors are impatient and want to consume early (in period 1). Resources being scarce, there are less funds available than projects of either type. In the absence of commitment problems, investors would put all their funds in early projects R_1 and capture the full return; Entrepreneurs would receive nothing. But financial intermediaries are needed to overcome commitment problems. In addition to the entrepreneur's commitment problem, specific collection skills are needed to transfer the return to the lender. As shown in Diamond/Rajan (2001, 2005), by issuing deposit contracts designed with a collective action problem (the risk of a bank run), bankers can credibly commit to use their collection skills to pass on to depositors the full amount received from entrepreneurs. So limited commitment motivates a role for banks as intermediaries.

(3) **Banks.** Due to their fragile structure, bankers are committed to pay out deposits as long as banks are not bankrupt. Holding capital (equity) can reduce the fragility of banks, but it allows bankers to capture a rent (assumed to be half of the surplus net of paying out depositors) and so lowers the amount of pledge able funds. Like entrepreneurs, bankers are indifferent between consuming early or late.

Banks offer deposit contracts. There is assumed to be perfect competition among bankers, so investors deposit their funds at those banks offering the highest expected return at the given market interest rate. Most of the time (see footnote 3), we assume that investors are able to monitor all bank's investment. So if, in a mixed strategy equilibrium, banks differ with respect to their investment strategy, the expected return from deposits must be the same across all banks.

Except for introducing two types of entrepreneurs, the structure of the model is essentially the same as the set up of Diamond/Rajan (2006). By assuming that depositors (investors) value consumption only at $t=1$, all relevant elements are captured in the most tractable way: at date 1, there is intertemporal liquidity trade with inelastic liquidity demand. Banks competing for funds at date 0 are forced to offer conditions which maximise expected consumption of investors at the given expected interest rates. Whereas Diamond/Rajan (2006) just present numerical examples for illustrating relevant cases, we fully characterise the type of equilibria as a function of parameter values. Furthermore, we derive endogenously the extent of financial fragility as a function of the parameter values.

As a reference point, section 3 analyses the case of pure idiosyncratic risk. It is shown that banks will choose their share of investment in safe projects such that all banks will be always solvent, given that there is liquid trading on the inter bank market. Section 4 introduces aggregate shocks. The outcome strongly depends on the probability of a bad aggregate shock occurring. If this probability is low, banks care only for the good state (proposition 1 a)) and accept the risk of failure with costly liquidation in the bad state. In contrast, banks play safe if the probability of a bad shock is very high (proposition 1 b)). For an intermediate range, however (proposition 2), financial intermediaries have an incentive to free ride on excess liquidity available in the good state. This leads to low liquidity in bad states. In the prevailing mixed-strategy equilibrium, depositors are worse off than if banks would coordinate on more liquid investment.

Section 5 analyses central bank intervention. With nominal bank contracts, monetary policy can help to prevent costly runs by injecting additional money before $t=1$. The real value of deposits will be reduced such that banks on the aggregate level are solvent despite the negative aggregate shock. Central bank intervention is welfare improving because of the inefficiency resulting from a bank run. But banks relying on central intervention will invest more in illiquid late projects. Liquidity provision improves the allocation in an incomplete market economy. Such a policy, however, may impede the role of money as a medium to facilitate ordinary transactions.

Section 6 examines the current debate on banks' equity requirements. We show that in the absence of aggregate risk such requirements only reduce banks' investments on safe projects as well as the investors' welfare. However, with the presence of aggregate risks equity holdings do help to absorb the aggregate shock and cushion the bad state. This is likely to improve the investors' welfare under certain conditions.

Section 7 concludes.

2 The Model - Basic Settings

2.1 Agents, Technologies and Preferences

There is a continuum of risk-neutral *investors* with unit endowment at $t = 0$ who want to consume at $t = 1$. They have only access to a storage technology with return 1, i.e. their wealth may be simply stored without perishing for future periods. As an alternative, they can lend their funds to finance profitable long term investments of entrepreneurs. Due to commitment problems, lending has to be done via financial intermediation.

There are two types of *entrepreneurs* which have ideas for projects: When funded, type i entrepreneurs can produce:

Type 1: Safe projects, yielding $R_1 > 1$ for sure early at date 1.

Type 2: Risky projects, yielding $R_2 > R_1 > 1$ either early at date 1 with probability p (and $pR_2 < R_1$), or late at date 2 with probability $1 - p$

Borrowing and lending is done via competitive and risk-neutral *banks* of measure N, who have no endowment at $t = 0$. Banks use the investor's funds (obtained via deposits or equity) to finance and monitor entrepreneurs projects. They have a special collection technology such that they can capture a constant share $0 < \gamma < 1$ of the project's return. The fragile banking structure allows them to commit to pass those funds which have been invested as deposits back to investors (see below). For funds obtained via equity, banks are able to capture a rent (assumed to be 1/2 of the captured return net of deposit claims).

Entrepreneurs and banks are indifferent between consumption at $t = 1$ or $t = 2$. Because only banks have the special skills in collecting deposits from investors and returns from entrepreneurs, entrepreneurs cannot contract with investors directly; instead, they can only get projects funded via bank loans.

Resources are scarce in the sense that there are more projects than aggregate endowment of investors. This excludes the possibility that entrepreneurs might bargain with banks on the level of γ.

2.2 Timing

There are 4 periods:

(1) $t = 0$

The banks offer deposit contract to investors, promising fixed payment d_0 in the future for each unit of deposit. The investors deposit their endowments if $d_0 \geq 1$. The banks then decide the share α of total funds to be invested in safe projects. Funded entrepreneurs receive loans and start their projects. d_0 and α are observable to all the agents, but p may be unknown at that date.

The *fixed payment deposit contract* has the following features:

a) Investors can claim a fixed payment d_0 for each unit of deposit at any date after $t = 0$;

b) Banks have to meet investors' demand with all resources available. If liquidity at hand is not sufficient, delayed projects have to be liquidated at a cost: Premature liquidation yields only c ($0 < c < 1 < \gamma R_1$) for each unit.

These contracts are adopted in the banking industry as a commitment mechanism. Since collecting returns from entrepreneurs requires specific skills, the bankers would have an incentive to renegotiate with lenders at $t = 1$ in order to exploit rents. So a standard contract would break down. As shown in Diamond /Rajan (2001), the debt contract can solve the problem of renegotiation: Whenever the investors anticipate a bank might not pay the promised amount, they will run and the bank's rent is completely destroyed by the costly liquidation. Therefore the banks will commit to the contract.

(2) $t = \frac{1}{2}$

At that intermediate date, p is revealed and so the investors can calculate the payment from the banks. If a banks resources are not sufficient to meet the deposit contract, i.e. the investors' expected average payment at $t = 1$ is $d_1 < d_0$ for each unit of deposit, all investors will run the bank already at $t = \frac{1}{2}$ in the attempt to be the first in the line, and so still being paid d_0. When a bank is run at $t = \frac{1}{2}$, she is forced to liquidate all projects immediately (even those which would be realized early) trying to satisfy the urgent demand of depositors – so in the case of a run, the bank will not be able to recover more than c from each project.

To concentrate on runs triggered by real shocks, we exclude self fulfilling panics: As soon as $d_1 \geq d_0$ investors are assumed never to run and to believe that the others don't run either.

(3) $t = 1$

If the investors didn't run in the previous period, they withdraw and consume. The banks collect a share γ from the early projects. But as long as entrepreneurs are willing to deposit their rents at $t = 1$ at banks, banks can pay out more resources to investors. Since early entrepreneurs retain the share $1-\gamma$ of the returns and they are indifferent between consumption at $t = 1$ or $t = 2$, the banks can borrow from them against the return of late projects at the market interest rate $r \geq 1$. r clears market by matching aggregate liquidity demand with aggregate liquidity supply. We assume that there is a perfectly liquid inter bank market at $t = 1$, so even if early entrepreneurs trade with other banks, the initial bank will be able to borrow the liquidity needed to refinance delayed projects as long as she is not bankrupt.

(4) $t = 2$

Banks collect return from late projects and repay the liquidity providers at $t = 1$. Both early and late entrepreneurs consume.

In the following sections we analyse the outcomes of the game in various situations.

3 Pure Idiosyncratic Shocks

As a baseline, consider the case in which p is deterministic and known to all the agents at $t = 0$. Equilibrium is characterised by the share α of funds banks choose to invest in safe projects and the interest rate r for deposits invested at $t = 1$. The outcome is captured in the following lemma:

LEMMA 1 *When p is deterministic, there exists a symmetric non-idle equilibrium of pure strategy in which*

1) *All the banks set* $\alpha_i(p,r)=\alpha^*(p,r)=\dfrac{\gamma\dfrac{1-p}{r}-(1-\gamma)p}{\gamma\dfrac{1-p}{r}+(1-\gamma)\left(\dfrac{R_1}{R_2}-p\right)}$, $\forall i\in[0,N]$;

2) *Interest rate* r *is determined by*

$$r\int_0^N(1-\gamma)\left[\alpha_iR_1+(1-\alpha_i)pR_2\right]di=\int_0^N\gamma(1-\alpha_i)(1-p)R_2di \text{ and } r\le\frac{R_2}{R_1}.$$

What's more, there exists no equilibrium of mixed strategies.

PROOF: See Appendix.

By LEMMA 1 multiple equilibria exist for all $1\le r\le\dfrac{R_2}{R_1}$. To make the analysis interesting, we introduce the following equilibrium selection criterion:

DEFINITION *An* ***optimal symmetric equilibrium of pure strategy*** *profile* $\alpha^*(p,r^*)$ *is given by*

(1) $r^*=\arg\max_r\kappa_i=\alpha_i(p,r)R_1+(1-\alpha_i(p,r))pR_2$;

(2) $\forall\alpha_i'(p,r^*)\ne\alpha^*(p,r^*)$ *with* $\alpha_{-i}(p,r^*)=\alpha^*(p,r^*)$,

$\kappa_i(\alpha^*(p,r^*))\ge\kappa_i(\alpha_i'(p,r^*),\alpha_{-i}(p,r^*))$ *in which* $-i\in[0,N]\setminus\{i\}$.

The optimal symmetric equilibrium is actually the Pareto-dominant equilibrium (as of Harsanyi / Selten (1988)) in which the banks collectively choose the strategy which maximizes their return. The banks choose to stay in such equilibrium unless there is opportunity for profitable unilateral deviation. See a similar argument (however, in different context) in Chen (1997).

LEMMA 2 *When p is deterministic, there exists a unique optimal symmetric equilibrium of pure strategy in which*

1) *All the banks set* $\alpha^*\left(p, r^*\right) = \dfrac{\gamma - p}{\gamma - p + (1-\gamma)\dfrac{R_1}{R_2}}$, $\forall i \in [0, N]$;

2) *Interest rate* $r^* = 1$.

From now on, denote $\alpha^*\left(p, r^*\right)$ by $\alpha(p)$ for simplicity.

PROOF: See Appendix.

Then if the risks are purely idiosyncratic, the equilibrium outcome is given by:

COROLLARY *When there are idiosyncratic risks such that for one bank i the probability p_i follows i.i.d. with pdf $f(p_i)$ with a non-empty support $\Omega \subseteq [0, \gamma]$, then there exists a unique optimal symmetric equilibrium of pure strategy in which*

1) *All the banks set* $\alpha\left(E[p_i]\right) = \dfrac{\gamma - E[p_i]}{\gamma - E[p_i] + (1-\gamma)\dfrac{R_1}{R_2}}$, $\forall i \in [0, N]$;

2) *Interest rate* $r^* = 1$.

This is pretty intuitive: As long as there are just idiosyncratic shocks, banks are always solvent via trade on the liquid inter bank market.

In the absence of aggregate risk, the optimal equilibrium can thus be characterised in a straightforward way. When there is only idiosyncratic risk, a share p of risky projects will always be realized early in the aggregate economy. The representative bank chooses the share α^* of funds invested in safe projects such that in period 1, it is able to pay out depositors and equity to all investors. Otherwise, the bank would be bankrupt and forced to liquidate late projects at high costs (liquidation gives an inferior return of $c < 1$).

So in the absence of aggregate risk, each bank invests in such a way that it is able to fulfill all claims of depositors at $t = 1$. At that date, early entrepreneurs pay back γR_j ($j \in \{1, 2\}$) to their bank. Being indifferent between consumption at $t=1$ and $t=2$, early entrepreneurs will also deposit all their own rents – the share of retained earnings $(1-\gamma)\,R_j$ – at safe banks. With a perfect liquid inter-bank market, unlucky banks with a high share of delayed projects are able to borrow loans from those banks which turn out to have a low share of delayed projects. So a representative bank is able to pay out at $t=1$ total resources available $\alpha\, R_1 + (1-\alpha)\, p\, R_2$ to depositors, when the market interest rate between $t=1$ and $t=2$ is $r^*=1$. As shown in LEMMA 2, $r^*=1$ supports the equilibrium giving depositors the highest expected return.

Depositors having a claim of $\gamma \cdot E\left[R(\alpha,r)\right] = \gamma\left[\alpha\left(\overline{p},r\right)R_1 + \left(1-\alpha\left(\overline{p},r\right)\right)R_2\right]$ per unit deposited, the total amount to be paid out via deposits at $r^*=1$ is $\gamma\,[\alpha\, R_1 + (1-\alpha)\, R_2]$. The representative bank will choose α^* such that at $t=1$, there are just enough resources available to pay out all depositors, taking into account that early entrepreneurs are reinvesting their rents at banks as deposits at $t=1$. The condition $\alpha\, R_1 + (1-\alpha)\, \overline{p}\, R_2 = \gamma\,[\alpha\, R_1 + (1-\alpha)\, R_2]$ gives as solution for α^* as a function of $\overline{p}$ (see FIGURE 1):

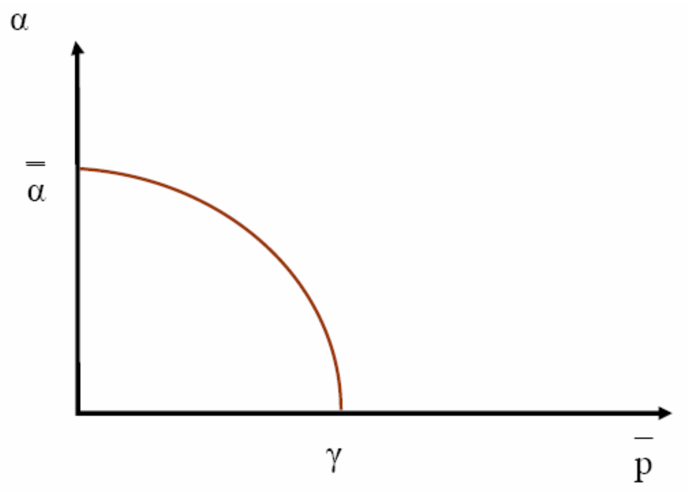

FIGURE 1 α^* as a function of $\overline{p}$

$$\alpha^*(\overline{p}) = \frac{\gamma - \overline{p}}{\gamma - \overline{p} + (1-\gamma)(R_1 / R_2)}$$

with $$\frac{\partial\, \alpha^*}{\partial\, \overline{p}} = \frac{-(1-\gamma)\,(R_1/R_2)}{[\gamma - \overline{p} + (1-\gamma)(R_1/R_2)]^2} < 0\,;\ \frac{\partial\, \alpha^*}{\partial\, R_2} < 0\,.$$

$$\alpha^* \in [0, \overline{\alpha}] \text{ with } \alpha^*(\overline{p}=0) = \overline{\alpha} = \frac{\gamma}{\gamma + (1-\gamma)(R_1 / R_2)} > \gamma \;;$$

$$\alpha^*(\overline{p}=\gamma) = 0 \,;\; \alpha^*(\gamma = 1) = 1$$

The higher $\overline{p}$ (the larger the share of early projects with a high payoff R_2), the lower the share of funds invested in projects of type R_1. If $\overline{p} > \gamma$, the representative bank would be solvent at $t = 1$ even when all funds were invested in the risky type of projects. Even if $\overline{p} = 0$, there will be some investment in projects with a high payoff R_2 as long as $\gamma < 1$. The reason is that all entrepreneurs, willing to wait until $t = 2$, can profit from higher returns of late projects. But for low interest rates $r^* = 1$, the investors as depositors also gain at least partly from the higher payoff of late projects, so $\alpha^*(\overline{p} = 0) > \gamma$. In the absence of a commitment problem (for $\gamma = 1$), however, there would be no funding of risky projects.

4 The Case of Aggregate Risk

The interesting case is the case of aggregate risk. Assume that p is now unknown to all the agents at $t = 0$ and realizes at $t = \frac{1}{2}$ as an aggregate risk. We assume that

(1) p can take just two possible values p_L or p_H with $0 < p_L < p_H < \gamma$;

(2) p_H realizes with probability π and p_L with probability $1 - \pi$.

In the presence of aggregate risk, a bank has several options available: The bank may just take care for provisions in the good state, choosing $\alpha^* = \alpha(p_H)$ and may take no precaution against the risk of a bank run in the bad state p_L. If so, the bank is run when p_L realizes and is forced to liquidate all projects. Obviously, this does not make sense if the probability of the bad state is high enough. Instead, the bank may increase the share of safe assets to $\alpha^* = \alpha(p_L)$ trying to prevent insolvency. If all banks would follow that strategy, there would be excess supply of liquidity in the good state p_H. This may give banks an incentive to free ride on the provision of liquidity by other banks, and a pure strategy equilibrium may not exist.[1] So a careful analysis of all cases is required. We will now show that there are 3 types of equilibria, depending on the probability π – the probability that a high share of early projects is realized.

A) PROPOSITION 1 a): If π is high enough (for $\pi \in [\overline{\pi}_2, 1]$, all banks will choose $\alpha^*(p_H)$. With that strategy, banks will be run at p_L, so depositors get only the return c if the share of early projects with high yields turns out to be unpleasantly low. All agents in the economy being risk neutral, it is more profitable for banks to take that risk into account in order to gain from the high returns in aggregate state p_H, as long as that event is not very likely.

B) PROPOSITION 1 b): If π is low enough (for $\pi \in [0, \overline{\pi}_1]$, all banks will choose $\alpha^*(p_L)$. In that case, banks will never be bankrupt, so they will be able to payout all depositors at $t = 1$ even if the share of delayed projects is high. But if the share of delayed projects is low (in the state p_H), there will be excess liquidity floating around at $t = 1$.

C) PROPOSITION 2: For some parameter constellations (for the intermediate range $\pi \in [\overline{\pi}_1, \overline{\pi}_2]$), banks will be tempted to free ride[2] on the excess liquidity in state p_H. These banks invest all their funds in the risky projects ($\alpha = 0$), trying to profit from the high returns available in case a large share of profitable projects happens to be realized early. The high expected returns in this case compensate depositors ex ante for the risk of getting just c in the other aggregate state of the world.

[1] Banks may also hold some equity in order to cushion shocks. We will discuss this in section 6 but ignore equity in this section.

[2] **Bhattacharya/ Gale** (1987) have already shown that there is free riding on liquidity provision when investors cannot monitor the amount of projects invested by the intermediaries. Footnote 3 confirms their argument in our context. But we derive a stronger result. We show that for an intermediate range of parameter values, even with perfect monitoring of banks, some banks have an incentive to free ride on liquidity in good states, giving rise to a mixed strategy equilibrium, resulting in excessively low liquidity in bad states. In the prevailing mixed-strategy equilibrium, depositors are worse off than if banks would coordinate on more liquid investment.

PROPOSITION 1 *Given* p_H *and* p_L, *and suppose that* α *'s are observable*[3] *to all investors:*

a) There is a unique optimal symmetric equilibrium of pure strategy such that all the banks set $\alpha^* = \alpha(p_H)$ *as soon as the probability of* p_H *satisfies* $\pi > \bar{\pi}_2 = \frac{\gamma \cdot E[R_L] - c}{\gamma \cdot E[R_H] - c}$, *in which* $E[R_s] = \alpha(p_s) R_1 + (1 - \alpha(p_s)) R_2,\ s \in \{H, L\}$ *; b) When* $0 \le \pi < \frac{\gamma \cdot E[R_L] - c}{\gamma \cdot E[R_H] - c} = \bar{\pi}_1$ *, there exists a unique optimal symmetric equilibrium of pure strategy such that all the banks set* $\alpha^* = \alpha(p_L)$.

PROOF: See Appendix.

The intuition behind PROPOSITION 1 is the following: When it is very unlikely that the low state realizes, i.e. π is very high, then the cost of a bank run is too small relative to the high return in the high state. So the best strategy for the banks is to exploit the maximum return from the high state and neglect the cost in the low state. On the contrary, when it is very likely that the low state realizes, then the cost of bank run is too high relative to the high return in the high state. Therefore the best strategy for the banks is to stick to the safest strategy and avoid the high cost in the low state. The interesting outcome takes place for intermediate π such that the cost of bank run is also intermediate and return from liquidity free-riding is sufficiently high in the high state:

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[3] This condition is crucial for $\pi \in [0, \bar{\pi}_2]$. If α 's were not observable to investors in this range, $\alpha(p_L)$ would fail to be a symmetric equilibrium of pure strategy. The reason is straight-forward: Suppose that all the banks coordinate and set $\alpha^* = \alpha(p_L)$, then there is always incentive for one single bank i to deviate and set $\alpha_i = \alpha(p_H)$ because she earns positive profit at p_H

$$\gamma\left[\alpha(p_H) R_1 + (1 - \alpha(p_H)) R_2\right] - \gamma\left[\alpha(p_L) R_1 + (1 - \alpha(p_L)) R_2\right] > 0,$$

and at p_L she is run with zero profit because of limited liability. In the end her expected profit is positive, which is larger than her peers who get zero profit because of perfect competition. Anticipating this, the banks would never coordinate to set $\alpha^* = \alpha(p_L)$.

PROPOSITION 2 *When* $\frac{\gamma \cdot E[R_L]-c}{\gamma R_2 - c} < \pi < \frac{\gamma \cdot E[R_L]-c}{\gamma \cdot E[R_H]-c}$, *there exists no optimal symmetric equilibrium of pure strategies. What's more, given* $p_H R_2 < R_1$ *and* c *not too high* ($c<1$) *there exists a unique equilibrium of mixed strategies such that for a representative bank:*

1) *With probability* θ *the bank chooses to be risky – she sets* $\alpha_r^* = 0$ *, and with probability* $1-\theta$ *to be safe – she sets* $\alpha_s^* > 0$ *;*

2) *Interest rates at states* p_H *and* p_L *are* $r_H > r_L > 1$ *;*

3) *At* $t=0$ *a risky bank offers a deposit contract with* $d_0^r = \gamma\left[p_H R_2 + \frac{(1-p_H)R_2}{r_H}\right]$ *and a safe bank with* $d_0^s = \gamma\left[\alpha_s^* R_1 + \left(1-\alpha_s^*\right)p_H R_2 + \frac{\left(1-\alpha_s^*\right)(1-p_H)R_2}{r_H}\right]$*;*

4) *Equal return condition:* $\kappa_r = \pi d_0^r + (1-\pi)c = d_0^s = \kappa_s$*;*

5) *Market clearing conditions:*

5a) At p_H *:* $\theta D_r + (1-\theta)D_s = \theta S_r + (1-\theta)S_s$ *, in which*

$$\begin{cases} D_r = d_0^r - \gamma p_H R_2, \\ D_s = d_0^s - \gamma\left[\alpha_s^* R_1 + \left(1-\alpha_s^*\right)p_H R_2\right], \\ S_r = (1-\gamma)p_H R_2, \\ S_s = (1-\gamma)\left[\alpha_s^* R_1 + \left(1-\alpha_s^*\right)p_H R_2\right]; \end{cases}$$

5b) At p_L *:* $r_L(1-\gamma)\left[\alpha_s^* R_1 + \left(1-\alpha_s^*\right)p_L R_2\right] = \gamma\left(1-\alpha_s^*\right)(1-p_L)R_2$*, i.e.* $\alpha_s^* = \alpha^*(p_L, r_L)$.

PROOF: See Appendix.

Though complicated, the intuition behind is still not difficult to see (To help the reader see the insight a numerical example is provided in Appendix 2). Suppose that we increase π from 0 where all the banks set $\alpha_i = \alpha(p_L)$. When π just gets higher than $\bar{\pi}_1$ free-riding on liquidity provision becomes profitable because

(1) The cost of bank run is no longer too high;

(2) At p_H the early entrepreneurs have excess liquidity supply. Therefore, an arbitrary bank i can free-ride and set her $\alpha_i' = 0$. By doing so she can trade liquidity at $t = 1$ from early entrepreneurs with high return from her late projects and promise $d_0' = \gamma R_2 > \gamma \cdot E[R_L] = d_0$ to the investors. The higher return in state p_H compensates the fact that she is surely run at p_L due to liquidity shortage.

But if every bank would behave as a free-rider, there would not be sufficient liquidity supply. So free-riders and safe banks must co-exist, i.e. the equilibrium is of mixed strategies.

The free-riding behaviour results in two consequences: (1) As more banks become free-riders, the interest rate r_H is bid higher; (2) The safe banks set lower $\alpha_s^* < \alpha(p_L)$ in order to cut down the opportunity cost of investing in safe projects. And in the end, r_H and α_s^* are adjusted such that depositors are indifferent between the two types of banks.

On the aggregate level the probability of being free-rider is determined by market clearing conditions for both states.

The resulting inefficiency is captured by the following corollary:

COROLLARY *For the equilibrium of mixed strategies defined by PROPOSITION 2, the banks are worse off than the case if they coordinate and choose* $\alpha_i = \alpha(p_L)$.

PROOF: The banks return is equal to $d_0^s = \kappa\left(\alpha^*(p_L, r_L)\right) < \kappa\left(\alpha(p_L)\right)$ by LEMMA 2, given the fact that $r_L > 1$.

Q.E.D.

5 Central Bank Intervention

Let us now consider the role of monetary policy. Suppose that central bank is now the fourth player in the game, and we make some slight changes to the original game in the following way:

(1) At $t = 0$ the banks provide *nominal* deposit contract to investors, promising a fixed nominal payment d_0 in the future. The central bank announces a minimum level $\underline{\alpha}$ of investment on safe projects required to be eligible for liquidity support in times of a crisis;

(2) At $t = \frac{1}{2}$ the banks decide whether to borrow liquidity from central bank. If yes, the central bank commits to provide liquidity for banks provided they fulfil the requirement $\underline{\alpha}$;

(3) At $t = 1$ the central bank injects liquidity by printing money at zero cost. It prefers to do so if the profit from money printing is non-negative;

(4) At $t = 2$ the banks repay the central bank by the collected return from surviving late projects.

For simplicity we assume that one unit of money is of equal value to one unit real good in payment. And the price level is determined by *cash-in-the-market* principle (Allen / Gale, 2005), i.e. the ratio of amount of liquidity (the sum of money and real goods) in the market to amount of real goods.

How will central bank intervention affect the outcome? In the model, it plays two roles. First, it helps to select the Pareto dominant equilibrium in the deterministic case. Second, it helps to prevent inefficient liquidation in the case of aggregate shocks.

If p is deterministic, interest rate will never exceed 1 because central bank always can borrow money to banks in order to get a share of return from late projects. Thus, due to competition the banks will maximize the real value of d_0 by setting $\alpha^* = \alpha(p,1)$, which is exactly the optimal equilibrium solution. Money plays a role as a device for equilibria selection, although it doesn't directly enter the market.

Much more interesting is the role of monetary policy in the case of aggregate shocks. By injecting liquidity in case of a crisis, the central bank prevents inefficient liquidation of early projects via bank runs, raising expected returns of banks choosing a risky strategy $\alpha^* = \alpha(p_H)$ when p_L is realized. So consider the case of aggregate shocks when π is high and the central bank sets $\underline{\alpha} = \alpha(p_H)$. In this case banks will set $\alpha^* = \alpha(p_H)$ and borrow liquidity from central bank only at p_L. Given this the investors will no longer run at p_L because they can only get c real goods plus $d_0 - c$ money for each unit of deposit. Instead if they wait till $t = 1$, they will get

$$\kappa[R_H \mid p_L] = \alpha(p_H)R_1 + (1-\alpha(p_H))\,p_L R_2 > c$$

real goods plus $d_0 - \kappa[R_H \mid p_L] = (1-\alpha(p_H))(p_H - p_L)R_2$ money, and they are better off by waiting.

Now the lower bound for $\alpha^* = \alpha(p_H)$ being the dominant strategy is shifted towards:

$$\overline{\pi}_2' = \frac{\gamma \cdot E[R_L] - \kappa[R_H | p_L]}{\gamma \cdot E[R_H] - \kappa[R_H | p_L]} < \frac{\gamma \cdot E[R_L] - c}{\gamma \cdot E[R_H] - c} = \overline{\pi}_2$$

So free-riding is partially deterred and the investors are better off with higher return,

$$\pi\gamma \cdot E[R_H] + (1-\pi)\kappa[R_H \mid p_L] > \pi\gamma \cdot E[R_H] + (1-\pi)c.$$

Injection of additional money before $t=1$ prevents costly runs. Obviously, banks relying on central intervention will invest more in illiquid late projects. The range of parameter values for which it is optimal to choose the risky strategy $\alpha^* = \alpha(p_H)$ is expanded. Nevertheless, liquidity provision improves the allocation in an incomplete market economy. Such a policy, however, may impede the role of money as a medium to facilitate ordinary transactions. This case will be analysed in future work.

6 The Role of Equity

Let us now introduce now capital requirements in the model. Under what conditions would it make sense to introduce equity requirements? It is easy to see that introducing equity will definitely reduce welfare in the absence of aggregate risk. Somewhat counterintuitive, capital requirements even reduces the share α invested in the safe project in that case. The reason is that with equity, bankers get a rent of $\frac{\gamma \cdot E[R] - d_0}{2}$, sharing the surplus over deposits equally with the equity holders. So investors providing funds in form of both deposits and equity to the banks will get out at $t=1$ just $\gamma \frac{E[R]}{1+k} < \gamma \cdot E[R]$. Since return at $t=2$ is higher than at $t=1$, bankers prefer to consume late, so the amount of resources needed at $t=1$ is lower in the presence of equity. Consequently, the share α will be reduced. Of course, banks holding no equity provide more attractive conditions for investors, so equity could not survive. This at first sight counterintuitive result simply demonstrates that there is no role (or rather only a welfare reducing role) for capital holding in the absence of aggregate risk.

But when there is aggregate risk, equity helps to absorb the aggregate shock. In the simple 2-state set up, equity holdings need to be just sufficient to cushion the bad state. So with equity, the bank will chose $\alpha^* = \alpha(p_H)$. The level of equity k needs to be so high that, given $\alpha^* = \alpha(p_H)$, the bank just stays solvent in the bad state – it is just able to payout the fixed claims of depositors, whereas all equity will be wiped out.

With equity k, the total amount that can be pledged to both depositors and equity in the good state is $\frac{1}{1+k} \gamma \cdot E\,[R(\alpha_H)]$ with claims of depositors being $D = \frac{1-k}{1+k} \gamma \cdot E\,[R(\alpha_H)]$ and equity

$EQ = \frac{k}{1+k}\gamma \cdot E\,[R(\alpha_H)]$. In the bad state, a marginally solvent bank can pay out to depositors $d_0 = \alpha\,(p_H)R_1 + (1-\alpha(p_H))\ p_L R_2$. So k is determined by the condition:

$$\frac{1-k}{1+k}\gamma \cdot E\,[R(\alpha_H)] = \alpha\,(p_H)R_1 + (1-\alpha(p_H))\ p_L R_2$$

k is decreasing in p_L: the higher p_L, the lower the equity k needed to stay solvent in the bad state. $k = 0$ for $p_L = p_H$.

For p_L close to p_H equity holding is superior to the strategy $\alpha^* = \alpha\left(p_H\right)$. That is if

$$d_0 \geq \gamma \cdot E\,[R(\alpha_H)]\,\pi + c\,(1-\pi).$$

7 Summary

To be written

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Appendix

1 Proofs

PROOF OF LEMMA 1: The proof is done by the following steps:

Claim 1: *Any non-idle equilibrium must be symmetric.*

Since the banks are competitive, therefore in equilibrium no bank is able to make strictly positive profit. Without restriction there exists a kind of equilibria in which some banks stay idle with zero profit by taking inferior strategies and getting no deposit at all. To make the results interesting, we exclude such equilibria throughout the paper.

As a direct conclusion, a representative bank i being active must achieve the same expected return $\kappa_i = \gamma\left[\alpha_i R_1 + (1-\alpha_i) pR_2\right] + \frac{\gamma(1-\alpha_i)(1-p)R_2}{r}$. Given equilibrium outcome r, all the banks should take the same α_i (here we don't require α_i be pure strategy).

Claim 2: *Any non-idle symmetric equilibrium of pure strategy takes the form stated in LEMMA 1.*

Consider a representative bank i with α_i. Her problem is to maximize her expected return, i.e.

$$\begin{aligned} \max_{\alpha_i}\ & \kappa_i = \gamma\left[\alpha_i R_1 + (1-\alpha_i) pR_2\right] + \frac{\gamma(1-\alpha_i)(1-p)R_2}{r} \\ s.t.\ & r\int_0^N (1-\gamma)\left[\alpha_i R_1 + (1-\alpha_i) pR_2\right] di = \int_0^N \gamma(1-\alpha_i)(1-p)R_2 di. \end{aligned}$$

The constraint above is just market clearing condition.

By symmetricity $r(\alpha_i) = \frac{\gamma(1-\alpha_i)(1-p)R_2}{(1-\gamma)\left[\alpha_i R_1 + (1-\alpha_i) pR_2\right]}$, then $\kappa_i = \alpha_i R_1 + (1-\alpha_i) pR_2$. Since $R_1 > pR_2$, α_i takes the maximum possible value which is given by $r(\alpha_i)$, solve to get

$$\alpha_i(p,r) = \frac{\gamma\frac{1-p}{r} - (1-\gamma)p}{\gamma\frac{1-p}{r} + (1-\gamma)\left(\frac{R_1}{R_2} - p\right)}.$$

Suppose that bank i deviates by setting $\alpha_i'(p,r) \neq \alpha^*(p,r)$. Then

1) If $\alpha_i'(p,r) < \alpha^*(p,r)$, $\kappa_i\left(\alpha_i'(p,r), \alpha_{-i}(p,r)\right) = \alpha_i' R_1 + (1-\alpha_i') pR_2 < \kappa_i\left(\alpha^*(p,r)\right)$;

2) If $\alpha_i'(p,r) > \alpha^*(p,r)$, given $r \leq \frac{R_2}{R_1}$

$$\begin{aligned}
\kappa_i\left(\alpha_i'(p,r), \alpha_{-i}(p,r)\right) &= \gamma\left[\alpha_i' R_1 + (1-\alpha_i') pR_2 + \frac{(1-\alpha_i')(1-p)R_2}{r}\right] \\
&< \gamma\left[\alpha^*(p,r) R_1 + \left(1-\alpha^*(p,r)\right) pR_2 + \frac{\left(1-\alpha^*(p,r)\right)(1-p)R_2}{r}\right] \\
&= \kappa_i\left(\alpha^*(p,r)\right).
\end{aligned}$$

Therefore no unilateral deviation is profitable.

<u>Claim 3</u>: *There exists no equilibrium of mixed strategies.*

Suppose that there exists an equilibrium of mixed strategies in which a representative bank i takes a mixed strategy σ_i with $\#\operatorname{supp}\sigma_i \geq 2$. Take two arbitrary elements $\alpha_i^1, \alpha_i^2 \in \operatorname{supp}\sigma_i$ and $\alpha_i^1 \neq \alpha_i^2$, given σ_{-i} and equilibrium outcome r the following equation must hold

$$\kappa_i\left(\alpha_i^1, \sigma_{-i}\right) = \kappa_i\left(\alpha_i^2, \sigma_{-i}\right)$$

meaning that $\alpha_i^1 = \alpha_i^2$. A contradiction.

Q.E.D.

PROOF OF LEMMA 2:

Since $\frac{\partial \kappa_i}{\partial r} = \frac{\partial \kappa_i}{\partial \alpha_i(p,r)} \frac{\partial \alpha_i(p,r)}{\partial r} < 0$ and $r \geq 1$, so $r^* = 1$ maximizes κ_i.

Suppose now bank i sets $\alpha_i'(p,r^*) \neq \alpha_i(p,r^*)$, then the liquidity she can borrow from early entrepreneurs is given by

$$\min\left[(1-\gamma)\left[\alpha_i'(p,r^*)R_1+\left(1-\alpha_i'(p,r^*)\right)pR_2\right],\frac{\gamma\left(1-\alpha_i'(p,r^*)\right)(1-p)R_2}{r^*}\right]$$

because of resource constraint. Then

1) For $\alpha_i'(p,r^*)>\alpha_i(p,r^*)$,

$$\begin{aligned}\kappa_i\left(\alpha_i'(p,r^*),\alpha_{-i}(p,r^*)\right)&=\gamma\left[\alpha_i'(p,r^*)R_1+\left(1-\alpha_i'(p,r^*)\right)R_2\right]\\&<\gamma\left[\alpha_i(p,r^*)R_1+\left(1-\alpha_i(p,r^*)\right)R_2\right]=\kappa_i\left(\alpha^*(p,r^*)\right);\end{aligned}$$

2) For $\alpha_i'(p,r^*)<\alpha_i(p,r^*)$,

$$\begin{aligned}\kappa_i\left(\alpha_i'(p,r^*),\alpha_{-i}(p,r^*)\right)&=\alpha_i'(p,r^*)R_1+\left(1-\alpha_i'(p,r^*)\right)pR_2\\&<\alpha_i(p,r^*)R_1+\left(1-\alpha_i(p,r^*)\right)pR_2=\kappa_i\left(\alpha^*(p,r^*)\right).\end{aligned}$$

So $\nexists\alpha_i'(p,r^*)\neq\alpha_i(p,r^*)$ such that $\kappa_i\left(\alpha^*(p,r^*)\right)<\kappa_i\left(\alpha_i'(p,r^*),\alpha_{-i}(p,r^*)\right)$.

Q.E.D.

PROOF OF PROPOSITION 1: By LEMMA 2 $\alpha(p_H)$ and $\alpha(p_L)$ maximize the banks' expected return at p_H and p_L respectively. The banks' expected return at p_H is higher than that at p_L because

$$\kappa\left(\alpha(p_H),p_H\right)=\gamma\left[\alpha(p_H)R_1+\left(1-\alpha(p_H)\right)R_2\right]=\gamma\cdot E[R_H]>\kappa\left(\alpha(p_L),p_L\right)=\gamma\cdot E[R_L].$$

However banks with $\alpha(p_H)$ are run at p_L and only get return of c, because

$$\kappa\left(\alpha(p_H),p_L\right)=\alpha(p_H)R_1+\left(1-\alpha(p_H)\right)p_LR_2<\alpha(p_H)R_1+\left(1-\alpha(p_H)\right)p_HR_2=\kappa\left(\alpha(p_H),p_H\right).$$

So the banks prefer $\alpha(p_H)$ to $\alpha(p_L)$ only if $\gamma\cdot E[R_H]\pi+(1-\pi)c>\gamma\cdot E[R_L]$, solve to get

$$\pi>\frac{\gamma\cdot E[R_L]-c}{\gamma\cdot E[R_H]-c}=\bar{\pi}_1.$$

When $\pi = 0$ the problem degenerates to deterministic case, so $\alpha^* = \alpha(p_L)$ is still unique optimal symmetric equilibrium of pure strategy.

When $0 < \pi < \overline{\pi}_1$ any strategic profile α^* in which all the banks set $\alpha^* \neq \alpha(p_L)$ cannot be optimal symmetric equilibrium of pure strategy:

1) For $\alpha^* \in (\alpha(p_H), \alpha(p_L))$, the maximum return one bank can obtain at p_L is $\alpha^* R_1 + (1-\alpha^*) p_L R_2 < \alpha(p_L) R_1 + (1-\alpha(p_L)) p_L R_2 = \kappa(\alpha(p_L))$, and the maximum return one bank can obtain at p_H is $\gamma\left[\alpha^* R_1 + (1-\alpha^*) R_2\right] > \gamma\left[\alpha(p_L) R_1 + (1-\alpha(p_L)) R_2\right] = \kappa(\alpha(p_L))$. Given this fact, the banks are run at p_L and only get an actual return of $\gamma\left[\alpha^* R_1 + (1-\alpha^*) R_2\right]\pi + (1-\pi)c$, but one can deviate by setting $\alpha_i = \alpha(p_H)$ making a higher expected return $\gamma\left[\alpha(p_H) R_1 + (1-\alpha(p_H)) R_2\right]\pi + (1-\pi)c$;

2) For $\alpha^* \in [0, \alpha(p_H))$ in which the banks are run at p_L (because $\alpha^* R_1 + (1-\alpha^*) p_H R_2 > \alpha^* R_1 + (1-\alpha^*) p_L R_2$), α^* is dominated by the optimal symmetric equilibrium of pure strategy $\alpha^* = \alpha(p_H)$ for deterministic p_H;

3) For $\alpha^* = \alpha(p_H)$, by PROPOSITION 1 α^* is dominated by $\alpha^* = \alpha(p_L)$;

4) For $\alpha^* \in (\alpha(p_L), 1]$ in which the banks survive at both states, α^* is dominated by $\alpha^* = \alpha(p_L)$ because $\gamma\left[\alpha^* R_1 + (1-\alpha^*) R_2\right] < \gamma\left[\alpha(p_L) R_1 + (1-\alpha(p_L)) R_2\right]$.

Now suppose that $\pi = \delta > 0$ and the banks still stick to $\alpha^* = \alpha(p_L)$. Then when p_H realizes with probability π, all early entrepreneurs have excess liquidity supply

$$\underbrace{(1-\gamma)\left[\alpha(p_L) R_1 + (1-\alpha(p_L)) p_H R_2\right]}_{\text{entrepreneurs' rent from ealy projects}} - \underbrace{\gamma(1-\alpha(p_L))(1-p_H) R_2}_{\text{early entrepreneurs' deposit in } t=1}$$
$$> (1-\gamma)\left[\alpha(p_L) R_1 + (1-\alpha(p_L)) p_L R_2\right] - \gamma(1-\alpha(p_L))(1-p_L) R_2 = 0.$$

Knowing this, one bank i can exploit this opportunity by setting $\alpha_i < \alpha\left(p_L\right)$ because all her liquidity shortage can be fulfilled by early entrepreneurs' deposit given $r^* = 1$. In this case $\alpha_i = 0$ maximizes her return at p_H, i.e. $\kappa_i\left(0, \alpha_{-i}\left(p_L\right)\right) = \gamma R_2 > \gamma \cdot E\left[R_L\right] = \kappa_i\left(\alpha_i\left(p_L\right)\right)$.

However any deviation $\alpha_i < \alpha\left(p_L\right)$ makes bank i run at p_L. Since α_i is observable by her depositors, her expected return for her investors is now

$$\gamma R_2 \pi + c\left(1-\pi\right).$$

Such deviation is profitable only if her expected return is higher than her peers, i.e.

$$\gamma R_2 \pi + c\left(1-\pi\right) > \gamma \cdot E\left[R_L\right] \Leftrightarrow \pi > \frac{\gamma \cdot E\left[R_L\right] - c}{\gamma R_2 - c}.$$

Otherwise all the banks would stick to $\alpha^* = \alpha\left(p_L\right)$.

Q.E.D.

PROOF OF PROPOSITION 2: The proposition is proved by construction.

<u>Claim 1</u>: *When* $\frac{\gamma \cdot E\left[R_L\right] - c}{\gamma R_2 - c} < \pi < \frac{\gamma \cdot E\left[R_L\right] - c}{\gamma \cdot E\left[R_H\right] - c}$, *there exists no optimal symmetric equilibrium of pure strategies.*

PROPOSITION 3 already shows that for $\frac{\gamma \cdot E\left[R_L\right] - c}{\gamma R_2 - c} < \pi < \frac{\gamma \cdot E\left[R_L\right] - c}{\gamma \cdot E\left[R_H\right] - c}$ there exists no optimal symmetric equilibrium of pure strategy because profitable unilateral deviation is always possible.

<u>Claim 2</u>: *If equilibrium of mixed strategies exist, the equilibrium can only have a two-point support* $\left\{\alpha_r^*, \alpha_s^*\right\}$ *such that one bank survives at both states by choosing* α_s^* *and survives at only one state by choosing* α_r^*.

Suppose that α_1 and α_2 ($\alpha_1 \neq \alpha_2$) are two arbitrary elements in the support of the mixed strategies equilibrium, r_H and r_L are the corresponding equilibrium interest rates at p_H and p_L respectively. One bank shall be indifferent between choosing α_1 and α_2.

Suppose that one bank survives at both states by choosing either α_1 and α_2. So her expected return should be the same for both strategies,

$$\gamma\left[\alpha_1 R_1 + \left(1-\alpha_1\right) p_H R_2 + \frac{\left(1-\alpha_1\right)\left(1-p_H\right) R_2}{r_H}\right] = \gamma\left[\alpha_2 R_1 + \left(1-\alpha_2\right) p_H R_2 + \frac{\left(1-\alpha_2\right)\left(1-p_H\right) R_2}{r_H}\right],$$

i.e. $\alpha_1 = \alpha_2$, a contradiction. Therefore there is at most one strategy by which one bank survives at both states.

Suppose that by choosing either α_1 and α_2 one bank survives at one state but is run in the other, so her expected return should be the same for both strategies:

$$\gamma\left[\alpha_1 R_1 + \left(1-\alpha_1\right) p_H R_2 + \frac{\left(1-\alpha_1\right)\left(1-p_H\right) R_2}{r_H}\right]\pi + \left(1-\pi\right)c$$
$$= \gamma\left[\alpha_2 R_1 + \left(1-\alpha_2\right) p_H R_2 + \frac{\left(1-\alpha_2\right)\left(1-p_H\right) R_2}{r_H}\right]\pi + \left(1-\pi\right)c,$$

i.e. $\alpha_1 = \alpha_2$, a contradiction.

Suppose that by choosing α_1 one bank survives at p_H and is run at p_L, and by choosing α_2 one bank survives at p_L and is run at p_H. This implies that

$$\gamma\left[\alpha_1 R_1 + \left(1-\alpha_1\right) p_H R_2 + \frac{\left(1-\alpha_1\right)\left(1-p_H\right) R_2}{r_H}\right] > \gamma\left[\alpha_1 R_1 + \left(1-\alpha_1\right) p_L R_2 + \frac{\left(1-\alpha_1\right)\left(1-p_L\right) R_2}{r_L}\right],$$

i.e. $p_H R_2 + \frac{\left(1-p_H\right) R_2}{r_H} > p_L R_2 + \frac{\left(1-p_L\right) R_2}{r_L}$, as well as

$$\gamma\left[\alpha_2 R_1 + \left(1-\alpha_2\right) p_H R_2 + \frac{\left(1-\alpha_2\right)\left(1-p_H\right) R_2}{r_H}\right] < \gamma\left[\alpha_2 R_1 + \left(1-\alpha_2\right) p_L R_2 + \frac{\left(1-\alpha_2\right)\left(1-p_L\right) R_2}{r_L}\right],$$

i.e. $p_H R_2 + \frac{(1-p_H)R_2}{r_H} < p_L R_2 + \frac{(1-p_L)R_2}{r_L}$, a contradiction.

Therefore there is at most one strategy by which one bank survives at one state and is run at the other.

Therefore the equilibrium profile of mixed strategies is supported by $\{\alpha_r^*, \alpha_s^*\}$ such that one bank survives at both states by choosing α_s^* and survives at only one state by choosing α_r^*.

<u>Claim 3</u>*: In such equilibrium, interest rates at states* p_H *and* p_L *are* $r_H > r_L > 1$.

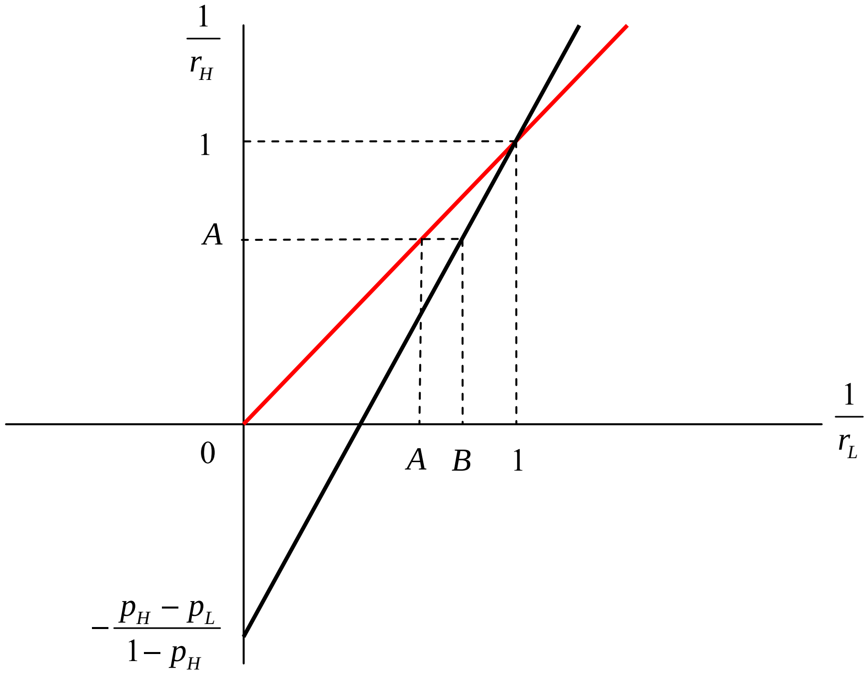
By choosing α_s^* one bank should have equal return at both states: $d_0^s = d_0^s(p_H) = d_0^s(p_L)$, i.e.

$$\gamma\left[\alpha_s^* R_1 + (1-\alpha_s^*) p_H R_2 + \frac{(1-\alpha_s^*)(1-p_H)R_2}{r_H}\right] = \gamma\left[\alpha_s^* R_1 + (1-\alpha_s^*) p_L R_2 + \frac{(1-\alpha_s^*)(1-p_L)R_2}{r_L}\right].$$

With some simple algebra this is equivalent to

$$\frac{1}{r_H} = \frac{1-p_L}{1-p_H}\frac{1}{r_L} - \frac{p_H - p_L}{1-p_H}.$$

Plot $\frac{1}{r_H}$ as a function of $\frac{1}{r_L}$:



The slope $\frac{1-p_L}{1-p_H} > 1$ and intercept $-\frac{p_H - p_L}{1-p_H} < 0$, and the line goes through $(1,1)$. But $r_H = r_L = 1$ cannot be equilibrium outcome here, because $\alpha(p_L)$ is dominant strategy in this case and subject to deviation. So whenever $r_H > 1$ (suppose $\frac{1}{r_H} = A$ in the graph), there must be $r_H > r_L > 1$ (because $\frac{1}{r_H} < \frac{1}{r_L} = B < 1$).

***<u>Claim 4</u>**: In such equilibrium, risky banks set $\alpha_r^* = 0$ and safe banks $\alpha_s^* > 0$. Risky banks promise $d_0^r = \gamma\left[p_H R_2 + \frac{(1-p_H)R_2}{r_H}\right]$ and are run at p_L; safe banks survive at both states by promising $d_0^s = \gamma\left[\alpha_s^* R_1 + (1-\alpha_s^*)p_k R_2 + \frac{(1-\alpha_s^*)(1-p_k)R_2}{r_k}\right]$ in which $k \in \{H, L\}$. Moreover, $\pi d_0^r + (1-\pi)c = d_0^s$.*

Since $\frac{(1-\alpha_s^*)(1-p_H)R_2}{r_H} < \frac{(1-\alpha_s^*)(1-p_L)R_2}{r_L}$, i.e. the safe banks get less liquidity from their early entrepreneurs at p_H, and also these early entrepreneurs have higher liquidity supply at

p_H (because $(1-\gamma)\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right) p_H R_2\right]>(1-\gamma)\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right) p_L R_2\right]$), therefore there must be excess liquidity supply from these early entrepreneurs at p_H and these excess liquidity supply must be absorbed at r_H by the risky banks. As a result, the risky banks survive at p_H by free-riding excess liquidity supply and are run at p_L.

At r_H by setting α_r^* the risky banks get a return of

$$d_0^r=\gamma\left[\alpha_r^* R_1+\left(1-\alpha_r^*\right) p_H R_2+\frac{\left(1-\alpha_r^*\right)\left(1-p_H\right) R_2}{r_H}\right].$$

Since the banks are risk-neutral the risky banks maximize the expression above by setting either $\alpha_r^*=0$ or $\alpha_r^*=1$ depending on all the other parameters. $\alpha_r^*=1$ is excluded because if so the banks become autarky and survive at both states. Therefore for $p_H R_2$ not too small and r_H not too big the risky banks maximize their return at r_H with $\alpha_r^*=0$. This determines d_0^r in the claim.

Moreover the expected return should be equal for both types of banks, $\pi d_0^r+(1-\pi) c=d_0^s$, to deter the deviation between types.

***<u>Claim 5</u>**: In such equilibrium, the strategy for the safe banks is given by $\alpha_s^*=\alpha^*\left(p_L, r_L\right)$, i.e.* $r_L(1-\gamma)\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right) p_L R_2\right]=\gamma\left(1-\alpha_s^*\right)\left(1-p_L\right) R_2$.

Since the risky banks are run and safe banks survive at p_L, given r_L the safe banks maximize their return by setting $\alpha_s^*=\alpha^*\left(p_L, r_L\right)$ by exhausting all liquidities provided by early entrepreneurs. By the proof of LEMMA 1 any unilateral deviation can only make lower return.

***<u>Claim 6</u>**: There exists proper solution of α_s^* for such equilibrium profile of mixed strategies.*

By $d_0^r \pi+(1-\pi) c=d_0^s$,

$$\gamma\left[p_H R_2+\frac{\left(1-p_H\right) R_2}{r_H}\right] \pi+(1-\pi) c=\gamma\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right) p_H R_2+\frac{\left(1-\alpha_s^*\right)\left(1-p_H\right) R_2}{r_H}\right] \cdots(A).$$

By $d_0^s = d_0^s\left(p_H\right) = d_0^s\left(p_L\right)$,

$$\gamma\left[\alpha_s^* R_1 + \left(1-\alpha_s^*\right) p_H R_2 + \frac{\left(1-\alpha_s^*\right)\left(1-p_H\right) R_2}{r_H}\right] = \alpha_s^* R_1 + \left(1-\alpha_s^*\right) p_L R_2 \cdots (B).$$

From (A) and (B), solve to get

$$\frac{\gamma\left(1-p_H\right) R_2}{r_H} = \frac{\alpha_s^* R_1 + \left(1-\alpha_s^*\right) p_L R_2 - \left(1-\pi\right) c - \pi\gamma p_H R_2}{\pi} \cdots (C).$$

Apply (C) into (B), by some simple algebra we get a quadratic equation of α_s^*

$$\left(R_1 - p_L R_2\right)\alpha_s^{*2} - \left[\pi\left(\gamma R_1 - c\right) - \left(p_L R_2 - c\right) + \left(1-\pi\right)\left(R_1 - p_L R_2\right)\right]\alpha_s^* - \left(p_L R_2 - c\right)\left(1-\pi\right) = 0 \cdots (D).$$

Define *LHS* of equation (D) as a function of α_s^*:

$f\left(\alpha_s\right) = \omega\alpha_s^{*2} + \phi\alpha_s^* + \varphi$, in which

$$\begin{cases} \omega = R_1 - p_L R_2 > 0, \\ \phi = -\left[\pi\left(\gamma R_1 - c\right) - \left(p_L R_2 - c\right) + \left(1-\pi\right)\left(R_1 - p_L R_2\right)\right], \\ \varphi = -\left(p_L R_2 - c\right)\left(1-\pi\right) < 0. \end{cases}$$

Since $\phi^2 - 4\omega\varphi > 0$, the quadratic equation has two real roots, denoted by $\alpha_{s,2}^* < \alpha_{s,1}^*$.

And by $\frac{\varphi}{\omega} < 0$ and $f\left(0\right) = \varphi < 0$, we know $\alpha_{s,2}^*\alpha_{s,1}^* < 0$, i.e. $\alpha_{s,2}^* < 0 < \alpha_{s,1}^*$.

Moreover we find that

$$\begin{aligned} f\left(1\right) &= \omega + \phi + \varphi \\ &= R_1 - p_L R_2 - \left[\pi\left(\gamma R_1 - c\right) - \left(p_L R_2 - c\right) + \left(1-\pi\right)\left(R_1 - p_L R_2\right)\right] - \left(p_L R_2 - c\right)\left(1-\pi\right) \\ &= \pi\left(1-\gamma\right) R_1 \\ &> 0, \end{aligned}$$

we know that $\alpha_{s,2}^* < 0 < \alpha_{s,1}^* < 1$.

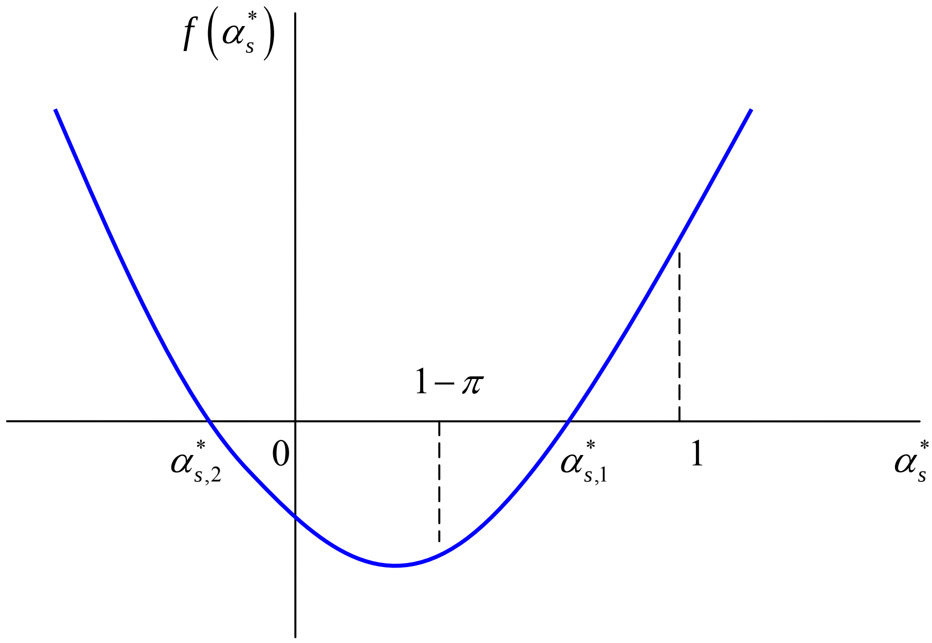
And again we can find that

$$
\begin{aligned}
f(1-\pi) &= (R_1 - p_L R_2)(1-\pi)^2 - \left[\pi(\gamma R_1 - c) - (p_L R_2 - c) + (1-\pi)(R_1 - p_L R_2)\right](1-\pi) \\
&\quad - (p_L R_2 - c)(1-\pi) \\
&= \left[-\pi(\gamma R_1 - c) + (p_L R_2 - c)\right](1-\pi) - (p_L R_2 - c)(1-\pi) \\
&= -\pi(\gamma R_1 - c)(1-\pi) \\
&< 0,
\end{aligned}
$$

we know that $\alpha_{s,2}^* < 0 < 1-\pi < \alpha_{s,1}^* < 1$.

This implies that in current settings, there always exists a plausible solution: $\alpha_{s,1}^* \in (1-\pi, 1)$.

All the arguments above can be captured by the following graph:



By equation (*A*)

$$
\gamma\left[p_H R_2 + \frac{(1-p_H)R_2}{r_H}\right]\pi + (1-\pi)c = \gamma\left[\alpha_s^* R_1 + (1-\alpha_s)p_H R_2 + \frac{(1-\alpha_s^*)(1-p_H)R_2}{r_H}\right],
$$

we already know that when $\pi = \bar{\pi}_2$, $\alpha_s^* = \alpha(p_L)$ and $r_H = 1$. When $\pi = \bar{\pi}_2 + \delta$, $\alpha_s \in (1-\pi, 1)$, then r_H has to be larger than 1 to make the equation still hold. From claim 3, this implies that $r_H > r_L > 1$.

Claim 7: *Given features described in previous claims, there exists no profitable unilateral deviation.*

Suppose that one bank i deviates by choosing $\alpha_i \neq \alpha_s^*$ and $\alpha_i \neq 0$. Then by doing so there are three possible consequences:

1) She survives at both states. But by claim 5 her return at p_L must be lower than d_0^s. If she survives at both states, she cannot promise $d_0^i \geq d_0^s$. Given this, no investor would deposit at all;

2) She survives at p_H but is run at p_L. Since $\alpha_i > 0$ by claim 4 her return at p_H must be lower than d_0^r;

3) She survives at p_L but is run at p_H. By 1) her return is $d_0^i < d_0^s$ at p_L and c at p_H. Her expected return is $d_0^i \pi + (1-\pi)c < d_0^s$.

Therefore strategic profile σ_i cannot be a profitable unilateral deviation such that σ_i contains $\alpha_i \neq \alpha_s^*$ and $\alpha_i \neq 0$ with probability $p \in (0,1]$.

And part 5) of the proposition is simply market clearing condition balancing aggregate liquidity supply and demand.

Q.E.D.

2 A Numerical Example for the Equilibrium of Mixed Strategies

Suppose that $p_H = 0.4$, $p_L = 0.3$, $\gamma = 0.6$, $R_1 = 2$, $R_2 = 4$, $c = 0.8$. Then

$$\alpha(p_H) = \frac{\gamma - p_H}{\gamma - p_H + (1-\gamma)R_1 / R_2} = \frac{0.6 - 0.4}{0.6 - 0.4 + (1-0.6)0.5} = \frac{1}{2},$$

$$\alpha(p_L) = \frac{\gamma - p_L}{\gamma - p_L + (1-\gamma)R_1 / R_2} = \frac{0.6 - 0.3}{0.6 - 0.3 + (1-0.6)0.5} = 0.6,$$

$$E[R_H] = \alpha(p_H)R_1 + (1-\alpha(p_H))R_2 = \frac{1}{2}\times 2 + \frac{1}{2}\times 4 = 3,$$
$$E[R_L] = \alpha(p_L)R_1 + (1-\alpha(p_L))R_2 = 0.6\times 2 + 0.4\times 4 = 2.8,$$

$$\bar{\pi}_2 = \frac{\gamma \cdot E[R_L] - c}{\gamma \cdot E[R_H] - c} = \frac{0.6\times 2.8 - 0.8}{0.6\times 3 - 0.8} = 0.88,$$

$$\bar{\pi}_1 = \frac{\gamma \cdot E[R_L] - c}{\gamma R_2 - c} = \frac{0.6\times 2.8 - 0.8}{0.6\times 4 - 0.8} = 0.55.$$

Take $\pi = 0.7 \in (\bar{\pi}_1, \bar{\pi}_2)$ and by $d_0^r \pi + (1-\pi)c = d_0^s$

$$\gamma\left[p_H R_2 + \frac{(1-p_H)R_2}{r_H}\right]\pi + (1-\pi)c = \gamma\left[\alpha_s^* R_1 + (1-\alpha_s^*)p_H R_2 + \frac{(1-\alpha_s^*)(1-p_H)R_2}{r_H}\right],$$

$$0.6\left[0.4\times 4 + \frac{0.6\times 4}{r_H}\right]\times 0.7 + 0.3\times 0.8 = 0.6\left[\alpha_s^*\times 2 + (1-\alpha_s)\times 0.4\times 4 + \frac{(1-\alpha_s^*)\times 0.6\times 4}{r_H}\right]\cdots(a).$$

By $d_0^s = d_0^s(p_H) = d_0^s(p_L)$,

$$\gamma\left[\alpha_s^* R_1 + (1-\alpha_s^*)p_H R_2 + \frac{(1-\alpha_s^*)(1-p_H)R_2}{r_H}\right] = \alpha_s^* R_1 + (1-\alpha_s^*)p_L R_2,$$

$$0.6\left[\alpha_s^*\times 2 + (1-\alpha_s^*)\times 0.4\times 4 + \frac{(1-\alpha_s^*)\times 0.6\times 4}{r_H}\right] = \alpha_s^*\times 2 + (1-\alpha_s^*)\times 0.3\times 4\cdots(b).$$

Solve equations (*a*) and (*b*) to get $\alpha_s^* = 0.47 < \alpha(p_H) < \alpha(p_L)$, $r_H = 1.519$.

And $d_0^s = \alpha_s^* R_1 + (1-\alpha_s^*)p_L R_2 = 1.576$, $d_0^r = \dfrac{d_0^s - (1-\pi)c}{\pi} = 1.908$.

Market clearing at p_L :

$$r_L\left(1-\gamma\right)\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right)p_L R_2\right]=\gamma\left(1-\alpha_s^*\right)\left(1-p_L\right)R_2,$$

$$r_L\times 0.4\left[0.47\times 2+0.53\times 0.3\times 4\right]=0.6\times 0.53\times 0.7\times 4,$$

solve to get $r_L=1.414$.

Market clearing at p_H :

$$D_r=d_0^r-\gamma p_H R_2=1.908-0.6\times 0.4\times 4=0.948,$$

$$D_s=d_0^s-\gamma\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right)p_H R_2\right]=1.576-0.6\left[0.47\times 2+0.53\times 0.4\times 4\right]=0.503,$$

$$S_r=\left(1-\gamma\right)p_H R_2=0.64,$$

$$S_s=\left(1-\gamma\right)\left[\alpha_s^* R_1+\left(1-\alpha_s^*\right)p_H R_2\right]=\left(1-0.6\right)\left[0.47\times 2+0.53\times 0.4\times 4\right]=0.715,$$

as well as

$$\theta D_r+\left(1-\theta\right)D_s=\theta S_r+\left(1-\theta\right)S_s,$$

solve to get $\theta=0.402$.